

Pointed homotopy and pointed lax homotopy of 2-crossed module maps

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Abstract

We address the (pointed) homotopy theory of 2-crossed modules (of groups), which are known to faithfully represent Gray 3-groupoids [31], with a single object, and also connected homotopy 3-types [15]. The homotopy relation between 2-crossed module maps will be defined in a similar way to Crans' 1-transfers between strict Gray functors [16], however being pointed, thus this corresponds to Baues' homotopy relation between quadratic module maps, treated in [4]. Despite the fact that this homotopy relation between 2-crossed module morphisms is not, in general, an equivalence relation, we prove that if \mathcal{A} and \mathcal{A}' are 2-crossed modules, with the underlying group F of \mathcal{A} being free (in short \mathcal{A} is free up to order one), then homotopy between 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$ yields, in this case, an equivalence relation. Furthermore, if a chosen basis B is specified for F , then we can define a 2-groupoid $\text{HOM}_B(\mathcal{A}, \mathcal{A}')$ of 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$, homotopies connecting them, and 2-fold homotopies between homotopies, where the latter correspond to (pointed) Crans' 2-transfers between 1-transfers.

We define a partial resolution $Q^1(\mathcal{A})$, for a 2-crossed module \mathcal{A} , whose underlying group is free, with a chosen basis, together with a projection map $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$, defining isomorphisms at the level of 2-crossed module homotopy groups. This resolution (proven to be part of a comonad in [25]) leads to a weaker notion of homotopies (lax homotopies) between 2-crossed module maps, which we fully develop and describe. In particular, given 2-crossed modules \mathcal{A} and \mathcal{A}' , there exists a 2-groupoid $\text{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$ of (strict) 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$, and their lax homotopies and lax 2-fold homotopies, leading to the question of whether the category of 2-crossed modules and strict maps can be enriched over the monoidal category Gray.

The associated notion of a (strict) 2-crossed module map $f: \mathcal{A} \rightarrow \mathcal{A}'$ to be a lax homotopy equivalence has the two-of-three property, and it is closed under retracts. This discussion leads to the issue of whether there exists a model category structure in the category of 2-crossed modules (and strict maps) where weak equivalences correspond to lax homotopy equivalences, and any free up to order one 2-crossed module is cofibrant.

Keywords Crossed module, 2-crossed module, quadratic module, homotopy 3-type, tricategory, Gray category, Peiffer lifting, simplicial group.

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1 Introduction and simplicial group background / context

Let $\mathcal{G} = (G_n, d_i^n, s_i^n; i \in \{0, 1, \dots, n\}, n = 0, 1, 2, \dots)$ be a simplicial group; [43, 26, 35, 17]. As usual, see for example [38, 24], we say that \mathcal{G} is free if each group G_n of n -simplices is a free group, with a chosen basis,

and these basis are stable under the degeneracy maps $s_i^n: G_n \rightarrow G_{n+1}$. Recall that the Moore complex [41, 42, 15] $N(\mathcal{G})$ of a simplicial group \mathcal{G} is given by the (normal) complex of groups $(\cdots \rightarrow A_n \xrightarrow{\partial_n} A_{n-1} \rightarrow \cdots \rightarrow A_0 = G_0)$, where:

$$A_n = \bigcap_{i=0}^{n-1} \ker(d_i^n),$$

and $\partial_n: A_n \rightarrow A_{n-1}$ is the restriction of the boundary map $d_n^n: G_n \rightarrow G_{n-1}$. We say that the Moore complex of \mathcal{G} has length n if the unique (possibly) non trivial components of $N(\mathcal{G})$ are $A_{n-1} \rightarrow A_{n-2} \rightarrow \cdots \rightarrow A_0$. (Here “length” correspond to the number of groups, rather than the number of arrows, which is the usual convention). Not surprisingly, this Moore complex has a lot of extra structure, defining what a hyper crossed complex is [12], which retains enough information to recover the original simplicial group, up to isomorphism. This contains two well known results, stating that the categories of simplicial groups with Moore complexes of length two and three (respectively) are equivalent to the categories of crossed modules and of 2-crossed modules of groups (respectively), see [41, 42, 15], the latter being exactly hyper crossed complexes of length two and three (respectively); we will go back to this issue below. Hyper crossed complexes therefore generalise both crossed modules and 2-crossed modules.

Looking at the last two stages of the Moore complex $N(\mathcal{G})$ of a simplicial group \mathcal{G} , namely $\partial = \partial_1: N_1(\mathcal{G}) \rightarrow N_0(\mathcal{G})$, one has an induced action of $N_0(\mathcal{G})$ on $N_1(\mathcal{G})$ by automorphisms, and also the action of $N_0(\mathcal{G})$ on itself by conjugation, and the boundary map $\partial: N_1(\mathcal{G}) \rightarrow N_0(\mathcal{G})$ preserves these actions; in other words one has a pre-crossed module ([10, 3, 4, 38]), called the pre-crossed module associated to the simplicial group \mathcal{G} .

The homotopy groups of a simplicial group \mathcal{G} are, by definition, given by the homology groups of its Moore complex $N(\mathcal{G})$ (which is a normal complex of, not necessarily abelian, groups). These correspond to the homotopy groups of the simplicial group \mathcal{G} seen as a simplicial set (despite the fact that $\pi_0(S)$, for S a simplicial set, is not in general a group but a set). Note that simplicial groups are Kan complexes, and therefore its homotopy groups are well defined [35, 17].

It is a fundamental result of Quillen [43, 26] that the category of simplicial groups is a model category, where weak equivalences are the simplicial group maps $f: \mathcal{G} \rightarrow \mathcal{G}'$, inducing isomorphisms at the level of homotopy groups, and fibrations are the simplicial groups maps $f: \mathcal{G} \rightarrow \mathcal{G}'$ whose induced map on Moore complexes $(f_i, i = 0, 1, 2, \dots): N(\mathcal{G}) \rightarrow N(\mathcal{G}')$ is surjective for all $i > 0$, and in particular any object is fibrant. Notice that, for $f = (f_i)$ to be a fibration, we do not impose that the induced map $f_0: A_0 = G_0 \rightarrow A'_0 = G'_0$ be surjective; however if f is a weak equivalence and a fibration then obligatory f_0 is surjective. Cofibrations are defined as being the maps that have the left lifting property with respect to all acyclic fibrations. In particular any free simplicial group is cofibrant [43, 24]. The pre-crossed module associated to a free simplicial group is of the form $F_1 \rightarrow F_0$ where F_0 is a free group and $F_1 \rightarrow F_0$ is a free pre-crossed module, [38]. Such a pre-crossed module is what is called in [38] a totally free pre-crossed module.

Let $\mathcal{S}\mathcal{G}$ denote the category of simplicial groups and $\mathcal{S}\mathcal{G}_n$ denote the full subcategory of simplicial groups with Moore complex of length n . The former is a reflexive subcategory of $\mathcal{S}\mathcal{G}$ and we denote the reflexion functor (the n -type or n^{th} -Postnikov section) by $P_n: \mathcal{S}\mathcal{G} \rightarrow \mathcal{S}\mathcal{G}_n$, a left adjoint to the inclusion functor $\mathcal{S}\mathcal{G}_n \rightarrow \mathcal{S}\mathcal{G}$. At the level of Moore complexes (A_m, ∂_m) is sent, via P_n , to (see [24]):

$$A_{n-1}/\partial(A_n) \rightarrow A_{n-2} \rightarrow A_{n-3} \rightarrow \cdots \rightarrow A_0.$$

This adjunction induces a closed model category structure on $\mathcal{S}\mathcal{G}_n$ where fibrations (weak equivalences) are the maps whose underlying simplicial group map is a fibration (weak equivalence), and cofibrations are the maps which have the left lifting property with respect to all acyclic fibrations; all of this is explained for example in [13]. Therefore the n -type functor P_n preserves cofibrations, [14]. This will give model category structures in the categories of crossed modules and of 2-crossed modules (of groups); [13, 14]. The case of crossed modules of groupoids is treated in [40, 37, 8], not appealing directly to simplicial group(oid) techniques.

A crossed module $(\partial: E \rightarrow G, \triangleright)$ is given by a group morphism $\partial: E \rightarrow G$, together with a left action \triangleright of G on E by automorphisms, so that ∂ preserves the actions, where G is given the adjoint action (thus $(\partial: E \rightarrow G, \triangleright)$ is a pre-crossed module), such that, furthermore, for each $e, f \in E$ the Peiffer pairing $\langle e, f \rangle \doteq (efe^{-1}) (\partial(e) \triangleright f^{-1}) \in E$ vanishes. The category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length two, where morphisms of crossed modules $(\partial: E \rightarrow G, \triangleright) \rightarrow (\partial': E' \rightarrow G', \triangleright')$ are given by chain maps $(\psi: E \rightarrow E', \phi: G \rightarrow G')$, preserving the actions. Crossed modules

form a model category [13, 37], where weak equivalences are the maps inducing isomorphisms on homotopy groups (the homology groups of the underlying complexes) and fibrations $(E \rightarrow G) \rightarrow (E' \rightarrow G')$ are the crossed module maps $(\psi: E \rightarrow E', \phi: G \rightarrow G')$, such that $\psi: E \rightarrow E'$ is surjective. The homotopy category of 2-crossed modules is equivalent to the homotopy category of pointed connected 2-types [34, 42, 3, 4] (where an n -type is a space X such that $\pi_i(X) = \{0\}$, if $i > n$). Since the reflexion functor $P_2: \mathcal{SG} \rightarrow \mathcal{SG}_2$ preserves cofibrations, a crossed module $(\partial: E \rightarrow G)$ is cofibrant when G is a free group, since in this case one can prove that there exists a free simplicial group (with Moore complex of length three) whose second Postnikov section is $(\partial: E \rightarrow G)$. The fact that, if $(\partial: E \rightarrow G, \triangleright)$ is a crossed module with G a free group, then it is cofibrant is directly proved for example in [40].

The 2-crossed modules were defined by Conduché [15], who proved that the category of 2-crossed modules is equivalent to the category of simplicial groups whose Moore complex has length three. A 2-crossed module is given by a complex of groups $(L \xrightarrow{\delta} E \xrightarrow{\partial} G)$, with a given action of G on E and L by automorphisms, such that $(\partial: E \rightarrow G)$ is a pre-crossed module, and also we have a map $\{, \}: E \times E \rightarrow L$ (the Peiffer lifting), lifting the Peiffer commutator map $<, >: E \times E \rightarrow E$, where, as before, $\langle e, f \rangle = (efe^{-1})(\partial(e) \triangleright f^{-1})$. This lifting has to satisfy very natural properties, satisfied by the Peiffer pairing itself. The quadratic modules, defined by Baues in [4] (being models for homotopy 3-types) are a special case of 2-crossed modules.

The category of 2-crossed modules is a Quillen model category, [13, 14], where a map $(\mu, \psi, \phi): (L \rightarrow E \rightarrow G) \rightarrow (L' \rightarrow E' \rightarrow G')$ is a fibration if, and only if, $\mu: L \rightarrow L'$ and $\psi: E \rightarrow E'$ each are surjective, and a weak equivalence if it induces isomorphisms of homotopy groups. Since the reflection functor $P_3: \mathcal{SG} \rightarrow \mathcal{SG}_3$ preserves cofibrations, one can see that any 2-crossed module $\mathcal{A} = (L \rightarrow E \rightarrow G)$, with $(E \rightarrow G)$ being a totally free pre-crossed module (in short \mathcal{A} is free up to order two), is cofibrant in this model category. Similar results are proved in [33], where an analogous model category structure for Gray categories is constructed, and it is proven that a Gray category is cofibrant if, and only if, its underlying sesquicategory [44] is free on a computad. We will go back to this issue below. Both 2-crossed modules and quadratic modules model the category of 3-types: pointed CW-complexes X such that $\pi_i(X) = \{0\}$ if $i > 3$, where the Whitehead products $\pi_2(X) \times \pi_2(X) \rightarrow \pi_3(X)$ are of course encoded in the Peiffer lifting. Quadratic modules are a reflexive subcategory of the category of 2-crossed modules; a reflection functor was constructed in [2].

Recall, see [16, 28, 27], that a Gray 3-category \mathcal{C} (or Gray enriched category) is a category enriched over the monoidal category of 2-categories, with the Gray tensor product. These can be given a more explicit definition, see [16, 31]. In particular, one has sets C_0, C_1, C_2 and C_3 of objects, morphisms, 2-morphisms and 3-morphisms, and several operations between them. In particular, objects, 1-morphisms and 2-morphisms of \mathcal{C} form a sesquicategory [44], a structure similar to a 2-category, but where the interchange law [31, 29] does not hold. Nevertheless, the interchange law holds up to a chosen tri-morphism: the “interchanger”. Gray 3-categories correspond to the strictest version of tri-categories, in the sense that any tricategory can be strictified to a tri-equivalent Gray-category, [28, 27]. For this reason Gray 3-categories are also called semistrict tri-categories.

Gray 3-groupoids can be defined analogously: there exists an inclusion of the category of 2-groupoids into the category of ω -groupoids [10] (equivalent to crossed complexes of groupoids), which has a left adjoint T (the cotruncation functor) similar to the 2-type functor. The tensor product of ω -groupoids is treated in [10, 9], and from now on called Brown-Higgins tensor product. By composing with the cotruncation functor yields a tensor product in the category of 2-groupoids (which is simply the restriction of the Gray tensor product of 2-categories to 2-groupoids), part of a monoidal closed structure. A Gray 3-groupoid is a groupoid enriched over this monoidal category of 2-groupoids. These are therefore Gray 3-categories where any i -cell ($i \geq 1$) is strictly invertible. It is folklore, and explicitly proven for example in [31, 7], that any 2-crossed module defines a Gray 3-groupoid with a single object (a “Gray 3-group”), and conversely. In particular the interchanger is derived from the Peiffer lifting in the given 2-crossed module.

The notion of a homotopy between ω -groupoid maps is treated in [9, 10, 6]. Considering n -fold homotopies between ω -groupoids maps defines an internal-hom “HOM” in the category of ω -groupoids, which, together with the Brown-Higgins tensor product of ω -groupoids, induces a monoidal closed structure. By applying the cotruncation functor this yields a monoidal closed structure in the category of 2-groupoids. If A and B are 2-groupoids then $\text{HOM}(A, B)$ is simply the 2-groupoid of strict functors $A \rightarrow B$, pseudo-natural transformations between them and their modifications. Considering 2-groups (2-groupoids with a single object) A and B , and pointed pseudo natural transformations and modifications, yields simply a groupoid with objects the maps $A \rightarrow B$, and morphisms their pointed pseudo-natural transformations. The latter groupoid is what we want to generalise for Gray 3-groups.

In [16], the notion of a pseudo-natural transformation (a 1-transfor) between strict functors of Gray 3-categories was addressed. Unlike the notion of pseudo-natural transformations between strict functors of 2-categories, this does not define (in general) an equivalence relation between Gray functors, the problem being that we cannot [16], in general, concatenate two 1-transfers, due to the lack of the interchange condition in Gray 3-categories. To have an equivalence relation between strict Gray functors one needs the full force of tricategories, leading to a less restrictive notion of 1-transfers, see [28, 27, 25]. In particular given Gray 3-categories A and B one has a Gray 3-category $\mathcal{HOM}(A, B)$ of strict Gray functors and their lax 1-, 2- and 3-transfers.

Given that strict 1-transfers between Gray 3-category maps $f, g: A \rightarrow B$ can be modeled by maps from the tensor product (of Gray categories [16]) $A \otimes I$ into B , where $A \otimes I$ is a cylinder object for A , the fact that two maps being related by a 1-transfer does not yield, in general, an equivalence relation is not at all surprising, given that in a model category such a construction would only be an equivalence relation, in general, if A were cofibrant and B were fibrant. As we have mentioned before A is cofibrant if, and only if, the underlying sesquicategory of A is free on a computad, [33].

In this article, we will analyse the notion of 1-transfers connecting Gray 3-groupoid maps, as well as 2-transfers connecting these, in context of 2-crossed modules. Similarly to [4, 21], we will restrict our discussion to the pointed case, and, since we are working with complexes of groups, we will name (strict) 1-transfers and 2-transfers as being homotopies and 2-fold homotopies. These are very similar to the usual notions of homotopies between chain complex maps, and 2-fold homotopies connecting them, however adapted to the non-abelian case.

To this end, given a 2-crossed module \mathcal{A}' , we will define a path-space $\mathcal{P}_*(\mathcal{A}')$ for it, together with two fibrations $\text{Pr}_0^{\mathcal{A}'}, \text{Pr}_1^{\mathcal{A}'}: \mathcal{P}_*(\mathcal{A}') \rightarrow \mathcal{A}'$. This will be a good path space [18], in the model category of 2-crossed modules. We use $\mathcal{P}_*(\mathcal{A}')$ to define the homotopy relation between 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$. This coincides with the notion defined ad-hoc in [21].

Let $\mathcal{A} = (L \rightarrow E \rightarrow G)$ and $\mathcal{A}' = (L' \rightarrow E' \rightarrow G')$ be 2-crossed modules (or more precisely the underlying complexes of them, since we also have actions \triangleright of G on E and L and a lifting $\{, \}: E \times E \rightarrow L$ of the Peiffer pairing, and the same for \mathcal{A}' .) A homotopy between the 2-crossed module maps $f, f': \mathcal{A} \rightarrow \mathcal{A}'$, explicitly $f = (\mu: L \rightarrow L', \psi: E \rightarrow E', \phi: G \rightarrow G')$ and $f' = (\mu': L \rightarrow L', \psi': E \rightarrow E', \phi': G \rightarrow G')$, is given by two set maps $s: G \rightarrow E'$ and $t: E \rightarrow L'$, satisfying appropriate properties (defining what we called a quadratic f -derivation), resembling the notion of homotopy between quadratic module maps, treated in [4]. For example, for all $g, h \in G$ we must have $s(gh) = \phi(h)^{-1} \triangleright s(g) s(h)$, thus $s: G \rightarrow E'$ is to be a derivation.

The notion of homotopy between 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$ is not an equivalence relation. This issue can be solved, for example, if we consider the case where the underlying pre-crossed module $(\partial: E \rightarrow G)$ of \mathcal{A} is totally free, thus \mathcal{A} is cofibrant in the model category of 2-crossed modules, which renders all objects fibrant.

What is surprising (and will be the main result of this paper) is that if $\mathcal{A} = (L \rightarrow E \rightarrow F)$ is a 2-crossed module, such that F is a free group (in short \mathcal{A} is free up to order one), then homotopy between 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$ defines an equivalence relation. Moreover, if a free basis B of F is specified, then we can define a groupoid, whose objects are the 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$ and the morphisms are their homotopies, represented as (for (s, t) a quadratic f -derivation):

$$f \xrightarrow{(f, s, t)} f'.$$

This process can be continued, to define a 2-groupoid $\text{HOM}_B(\mathcal{A}, \mathcal{A}')$, with objects being the 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$, the morphisms being the homotopies between them, and the 2-morphisms being 2-fold homotopies between homotopies. We will present very detailed calculations. One of the first issues this result raises is whether the category of free up to order one 2-crossed modules (and strict maps, preserving chosen basis) can be enriched over the monoidal category Gray, in other words whether it is a Gray 3-category.

Consider two 2-crossed modules:

$$\mathcal{A} = \left(L \xrightarrow{\delta} E \xrightarrow{\partial} F, \triangleright, \{, \} \right) \text{ and } \mathcal{A}' = \left(L' \xrightarrow{\delta} E' \xrightarrow{\partial} G', \triangleright, \{, \} \right),$$

such that F is a free group, with a chosen basis B . If the quadratic f -derivation $(s: F \rightarrow E', t: E \rightarrow L')$ connects $f: \mathcal{A} \rightarrow \mathcal{A}'$ and $f': \mathcal{A} \rightarrow \mathcal{A}'$ and $(s': F \rightarrow E', t': E \rightarrow L')$ connects $f': \mathcal{A} \rightarrow \mathcal{A}'$ and $f'': \mathcal{A} \rightarrow \mathcal{A}'$, diagrammatically:

$$f \xrightarrow{(f, s, t)} f' \text{ and } f' \xrightarrow{(f', s', t')} f'',$$

then we explicitly construct a quadratic f -derivation $(s \otimes s', f \otimes f')$, such that:

$$f \xrightarrow{(f, s \otimes s', t \otimes s'')} f''.$$

This concatenation “ \otimes ” of homotopies is associative and it has inverses. The derivation $s \otimes s': F \rightarrow E'$ is the unique derivation $F \rightarrow E'$ which, on the chosen basis B of F , has the form $b \mapsto s(b)s'(b)$. In the case when $(\partial: E' \rightarrow G')$ is a crossed module, and the Peiffer lifting of \mathcal{A}' is trivial, we have that $(s \otimes s')(g) = s(g)s'(g)$ for each $g \in F$. Otherwise one has a map $\omega^{(s, s')}: F \rightarrow L'$, measuring the difference between $s \otimes s'$ and the pointwise product of s and s' ; namely we have:

$$(s \otimes s')(g) = s(g) s'(g) \delta(\omega^{(s, s')}(g))^{-1}, \text{ for each } g \in F.$$

This map $\omega^{(s, s')}: F \rightarrow L'$ has a prime importance in the construction of the concatenation of homotopies.

Let G be a group. The free group on the underlying set of G is denoted by $\mathcal{F}^{\text{group}}(G)$. The set inclusion $G \rightarrow \mathcal{F}^{\text{group}}(G)$ is written as $g \in G \mapsto [g] \in \mathcal{F}^{\text{group}}(G)$. We have the obvious basis $[G] = \{[g], g \in G\}$ of $\mathcal{F}^{\text{group}}(G)$. Consider the obvious projection group map $p: \mathcal{F}^{\text{group}}(G) \rightarrow G$, thus $p([g]) = g$. There exists a very natural partial resolution functor Q^1 , from the category of 2-crossed modules to itself, which, to a 2-crossed module $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$, associates the 2-crossed module:

$$Q^1(\mathcal{A}) = \left(L \xrightarrow{\delta'} E_{\partial} \times p\mathcal{F}^{\text{group}}(G) \xrightarrow{\partial'} \mathcal{F}^{\text{group}}(G), \triangleright, \{, \} \right),$$

where:

$$E_{\partial} \times p\mathcal{F}^{\text{group}}(G) = \{(e, u) \in E \times \mathcal{F}^{\text{group}}(G) : \partial(e) = p(u)\}.$$

Therefore $Q^1(\mathcal{A})$ is free up to order one. Moreover there is a projection $\text{proj}: (r, q, p): Q^1(\mathcal{A}) \rightarrow \mathcal{A}$ which yields isomorphisms at the level of homotopy groups. It has the form:

$$\begin{array}{ccccc} & L & \xrightarrow{\delta'} & E_{\partial} \times p\mathcal{F}^{\text{group}}(G) & \xrightarrow{\partial'} & \mathcal{F}^{\text{group}}(G) \\ \text{proj} = & r \downarrow & & q \downarrow & & p \downarrow \\ & L & \xrightarrow{\delta} & E & \xrightarrow{\partial} & G. \end{array} \quad (1)$$

where $r = \text{id}$ and $q(e, u) = e$. Therefore $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$ is surjective and a weak equivalence (thus an acyclic fibration in the model category of 2-crossed modules).

The map $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$, resembling a cofibrant replacement, is proven to be part of a comonad in [25]. Its co-Kleisli category [36] leads to weaker notions of maps $\mathcal{A} \rightarrow \mathcal{A}'$ between 2-crossed modules [23, 25], and of homotopies between strict 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$, as well as of their 2-fold homotopies, yielding a 2-groupoid $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$, of strict maps $\mathcal{A} \rightarrow \mathcal{A}'$, lax homotopies and lax 2-fold homotopies, which we fully describe,

Let us be specific. Let \mathcal{A} and \mathcal{A}' be 2-crossed modules. Let $\text{hom}(\mathcal{A}, \mathcal{A}')$ denote the set of 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$. Given that $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$ is surjective, the map $f \in \text{hom}(\mathcal{A}, \mathcal{A}') \mapsto f \circ \text{proj} \in \text{hom}(Q^1(\mathcal{A}), \mathcal{A}')$ is an injection. A 2-crossed module map $Q^1(\mathcal{A}) \rightarrow \mathcal{A}'$ is said to be strict if it factors (uniquely) through $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$. Then we define $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$ as being the full sub-2-groupoid of $\text{HOM}_{[G]}(Q^1(\mathcal{A}), \mathcal{A}')$, with objects being the strict maps $Q^1(\mathcal{A}) \rightarrow \mathcal{A}'$. After presenting $Q^1(\mathcal{A})$ combinatorially (by generators and relations), we will give a fully combinatorial description of lax homotopies between strict 2-crossed module maps, and their lax 2-fold homotopies, therefore explicitly constructing the 2-groupoid $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$.

Lax homotopies between strict 2-crossed module maps behave well with respect to composition by strict 2-crossed module maps. Therefore it is natural to conjecture that the category of 2-crossed modules, strict 2-crossed module maps, lax homotopies and their 2-fold homotopies is a Gray 3-category.

We say that a 2-crossed module map $f: \mathcal{A} \rightarrow \mathcal{A}'$ is a lax homotopy equivalence if there exists a 2-crossed module map $g: \mathcal{A}' \rightarrow \mathcal{A}$ such that $f \circ g$ and $g \circ f$ each are lax homotopic to $\text{id}_{\mathcal{A}'}$ and $\text{id}_{\mathcal{A}}$, respectively. Since we can concatenate lax homotopies between 2-crossed module maps, the class of lax homotopy equivalences has the two-of-three property; [18]. Given that we can compose lax homotopies with strict 2-crossed module maps, any retract of a lax homotopy equivalence is a lax homotopy equivalence. All of this discussion leads

to the issue of whether there exists a model category structure in the category of 2-crossed modules (different from the one already referred to, which has as cofibrant objects the retracts of free up to order two 2-crossed modules) where weak equivalences correspond to lax homotopy equivalences, and where free up to order one 2-crossed modules are cofibrant. This would resemble both the Strøm model category structure [45] in the category of topological spaces, where the homotopy relation between maps is the usual relation of homotopy and all objects are fibrant and cofibrant, and also the Strøm like model category structure in the category of 2-categories, constructed in [19], where all 2-categories are fibrant and cofibrant, and the homotopy relation between maps correspond to them being related by a pseudo-natural transformation. This model structure in the category of 2-categories is different from the one defined in [32], where a 2-category is cofibrant if and only if its underlying category is free.

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2 Preliminaries on pre-crossed modules and 2-crossed modules

All actions of a group G on a set X are to associate the identity map id_X of X to the identity 1_G of G .

2.1 Pre-crossed modules and crossed modules

Definition 1 (Pre-crossed module) A pre-crossed module $(\partial: E \rightarrow G, \triangleright)$ is given by a group morphism $\partial: E \rightarrow G$, together with a left action \triangleright of G on E by automorphisms, such that the following relation, called “first Peiffer relation”, holds:

$$\partial(g \triangleright e) = g e g^{-1}, \text{ for each } g \in G \text{ and each } e \in E.$$

A crossed module $(\partial: E \rightarrow G, \triangleright)$ is a pre-crossed module satisfying, further, the “second Peiffer relation”:

$$\partial(x) \triangleright y = x y x^{-1}, \text{ for each } x, y \in E.$$

Note that in a crossed module $(\partial: E \rightarrow G, \triangleright)$ the subgroup $\ker(\partial) \subset E$ is central in E .

Let $(\partial: E \rightarrow G, \triangleright)$ be a pre-crossed module. Given $x, y \in E$, their *Peiffer commutator* is given by

$$\langle x, y \rangle = (x y x^{-1}) (\partial(x) \triangleright y^{-1}).$$

Thus a pre-crossed module is a crossed module if, and only if, all of its Peiffer commutators are the identity of E . In any pre-crossed module it holds that for each $x, y \in E$ (and where 1_G is the identity of G):

$$\partial(\langle x, y \rangle) = 1_G.$$

A morphism $f = (\psi, \phi)$ between the pre-crossed modules $(\partial: E \rightarrow G, \triangleright)$ and $(\partial': E' \rightarrow G', \triangleright')$ is given by a pair of group morphisms $\psi: E \rightarrow E'$ and $\phi: G \rightarrow G'$ making the diagram:

$$\begin{array}{ccc} E & \xrightarrow{\partial} & G \\ \psi \downarrow & & \downarrow \phi \\ E' & \xrightarrow{\partial'} & G' \end{array}$$

commutative, and such that

$$\psi(g \triangleright e) = \phi(g) \triangleright' \psi(e), \text{ for each } e \in E \text{ and } g \in G.$$

Morphisms of crossed modules are defined analogously, and therefore the category of crossed modules is a full subcategory of the category of pre-crossed modules.

Example 2 (The underlying group functor Gr_0) There is an underlying group functor Gr_0 sending a pre-crossed module $(E \rightarrow G, \triangleright)$ to the group G . This has a right adjoint R sending a group G to the crossed module $(\text{id}: G \rightarrow G, \text{ad})$, where we consider the adjoint action ad of G on G . The unit of this adjunction, yields a pre-crossed module map $\eta_G = (\partial, \text{id}): (\partial: E \rightarrow G, \triangleright) \rightarrow (\text{id}: G \rightarrow G, \text{ad}) \doteq \eta(\partial: E \rightarrow G, \triangleright)$, for each pre-crossed module $\mathcal{G} = (\partial: E \rightarrow G, \triangleright)$.

Definition 3 (The principal group functor Gr_1) There is a principal group functor Gr_1 from the category of pre-crossed modules to the category of groups, sending a pre-crossed module $(E \rightarrow G, \triangleright)$ to E .

2.2 Definition of 2-crossed modules and elementary properties. The secondary action \triangleright' of a 2-crossed module

We will follow the conventions of [15, 22] for the definition of a 2-crossed module. Important references on 2-crossed modules are [38, 31, 7, 41].

Definition 4 (2-crossed module) A 2-crossed module (of groups) is given by a chain complex of groups:

$$L \xrightarrow{\delta} E \xrightarrow{\partial} G$$

together with left actions \triangleright , by automorphisms, of G on L and E (and on G by conjugation), and a G -equivariant function $\{, \} : E \times E \rightarrow L$ (called the Peiffer lifting). Here G -equivariance means:

$$g \triangleright \{e, f\} = \{g \triangleright e, g \triangleright f\}, \text{ for each } g \in G \text{ and } e, f \in E.$$

These are to satisfy:

1. $L \xrightarrow{\delta} E \xrightarrow{\partial} G$ is a chain complex of G -modules (in other words ∂ and δ are G -equivariant and $\partial \circ \delta = 1$.)
2. $\delta(\{e, f\}) = \langle e, f \rangle$, for each $e, f \in E$. Recall that $\langle e, f \rangle = (efe^{-1})(\partial(e) \triangleright f^{-1})$.
3. $[l, k] = \{\delta(l), \delta(k)\}$, for each $l, k \in L$. Here $[l, k] = lkl^{-1}k^{-1}$.
4. $\{\delta(l), e\} \{e, \delta(l)\} = l(\partial(e) \triangleright l^{-1})$, for each $e \in E$ and $l \in L$.
5. $\{ef, g\} = \{e, fgf^{-1}\} \partial(e) \triangleright \{f, g\}$, for each $e, f, g \in E$.
6. $\{e, fg\} = \{e, f\} (\partial(e) \triangleright f) \triangleright' \{e, g\}$, where $e, f, g \in E$.

Here we have put:

$$e \triangleright' l = l \{\delta(l)^{-1}, e\}, \text{ where } l \in L \text{ and } e \in E. \quad (2)$$

The following is very well know; see [15, 41].

Lemma 5 (Secondary action \triangleright' of a 2-crossed module) Let $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be a 2-crossed module. The map $(e, l) \in G \times E \mapsto e \triangleright' l$ of equation (2) is a left action of E on L by automorphisms, called the “secondary action of \mathcal{G} ”. Together with the map $\delta : L \rightarrow E$, the action \triangleright' defines a crossed module.

Let us give details. A complete proof that $l \in L \mapsto e \triangleright' l \in L$, for an $e \in E$, satisfies $e \triangleright' (lk) = (e \triangleright' l) (e \triangleright' k)$, for each $l, k \in L$, is in [15, page 163]. In particular $e \triangleright' 1_L = 1_L$ for each $e \in E$; here 1_L denotes the identity of the group L . Therefore, from the explicit formula for \triangleright' , we must have, for each $e \in E$, that $\{1_E, e\} = 1_L$, where 1_E is the identity of E . Also, if $e \in E$, then:

$$\{\delta(1_L), e\} \{e, \delta(1_L)\} = 1_L \partial(e) \triangleright 1_L^{-1} = 1_L,$$

thus also $\{e, 1_E\} = 1_L$ for each $e \in E$. We have in particular proven that given any $e \in E$:

$$\{e, 1_E\} = \{1_E, e\} = 1_L. \quad (3)$$

Let us now see that \triangleright' defines an action of E on L . Given $e, f \in E$ and $l \in L$ we have:

$$\begin{aligned} (ef) \triangleright' l &= l \{\delta(l)^{-1}, ef\} = l \{\delta(l)^{-1}, e\} (\partial(\delta(l)) \triangleright e) \triangleright' \{\delta(l)^{-1}, f\} = l \{\delta(l)^{-1}, e\} e \triangleright' \{\delta(l)^{-1}, f\} \\ &= (e \triangleright' l) (e \triangleright' \{\delta(l)^{-1}, f\}) = e \triangleright' (l \{\delta(l)^{-1}, f\}) = e \triangleright' (f \triangleright' l). \end{aligned}$$

Also, if $l \in L$:

$$1_E \triangleright' l = l \{\delta(l)^{-1}, 1_E\} = l.$$

That, together with the map $\delta : L \rightarrow E$, the action \triangleright' defines a crossed module follows from axioms 2 and 3 of definition 4. Note that in particular it follows that $\ker(\delta) \subset L$ is central in L .

We also have:

$$\{\delta(l)^{-1}, e\}^{-1} l^{-1} = (e \triangleright' l)^{-1} = e \triangleright' l^{-1} = l^{-1} \{\delta(l), e\}, \quad (4)$$

$$e \triangleright' l = \{\delta(l), e\}^{-1} l, \quad (5)$$

$$\partial(e) \triangleright l = (e \triangleright' l) \{e, \delta(l)^{-1}\}, \quad (6)$$

$$\partial(e) \triangleright l = \{e, \delta(l)\}^{-1} e \triangleright' l. \quad (7)$$

Therefore $\partial(a) \triangleright l = a \triangleright' l$ if $a \in E$ and $l \in \ker \delta$; equation (3). For each $a, b, c \in E$ we have:

$$a \triangleright' \{b, c\} = \partial(a) \triangleright \{b, c\} \{a, \langle b, c \rangle^{-1}\}^{-1} = \{\partial(a) \triangleright b, \partial(a) \triangleright c\} \{a, (\partial(b) \triangleright c) b c^{-1} b^{-1}\}^{-1}. \quad (8)$$

A morphism $f = (\mu, \psi, \phi)$ between the 2-crossed modules $\mathcal{A}_1 = (L_1 \rightarrow E_1 \rightarrow G_1, \triangleright_1, \{, \}_1)$ and $\mathcal{A}_2 = (L_2 \rightarrow E_2 \rightarrow G_2, \triangleright_2, \{, \}_2)$ is given by group morphisms $\mu: L_1 \rightarrow L_2, \psi: E_1 \rightarrow E_2$ and $\phi: G_1 \rightarrow G_2$, defining a chain map between the underlying complexes, such that, for each $e, f \in E, g \in G$ and $k \in L$:

$$\mu(\{e, f\}_1) = \{\psi(e), \psi(f)\}_2, \quad \mu(g \triangleright_1 k) = \phi(g) \triangleright_2 \mu(k) \text{ and } \psi(g \triangleright_1 e) = \phi(g) \triangleright_2 \psi(e).$$

The set of 2-crossed module morphisms $\mathcal{A}_1 \rightarrow \mathcal{A}_2$ is denoted by $\text{hom}(\mathcal{A}_1, \mathcal{A}_2)$.

For a proof of the following lemma we refer to [21, 22]. Compare with (14) and (15).

Lemma 6 *In a 2-crossed module $(L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \}_)$ we have, for each $e, f, g \in E, a \in G$ and $k \in L$:*

$$\{ef, g\} = (e \triangleright' \{f, g\}) \{e, \partial(f) \triangleright g\} \text{ and } \{e, fg\} = ((efe^{-1}) \triangleright' \{e, g\}) \{e, f\}. \quad (9)$$

$$a \triangleright (e \triangleright' k) = (a \triangleright e) \triangleright' (a \triangleright k). \quad (10)$$

$$\{e, f\}^{-1} = \partial(e) \triangleright \{e^{-1}, efe^{-1}\}, \quad \{e, f\}^{-1} = (efe^{-1}) \triangleright' \{e, f^{-1}\}, \quad (11)$$

$$\{e, f\}^{-1} = (\partial(e) \triangleright f) \triangleright' \{e, f^{-1}\}, \quad \{e, f\}^{-1} = e \triangleright' \{e^{-1}, \partial(e) \triangleright f\}. \quad (12)$$

$$a \triangleright' \{n, m\} = \{a n \partial(n^{-1}) \triangleright a^{-1}, (\partial(n^{-1}) \triangleright a) m (\partial(n^{-1}) \triangleright a^{-1})\}, \quad (13)$$

$$\{e, fg\} = \{ef \partial(e^{-1}) \triangleright (ef^{-1}e^{-1}), \partial(e) \triangleright (efe^{-1}) g \partial(e^{-1}) \triangleright (ef^{-1}e^{-1})\} \{e, f\}, \quad (14)$$

$$\{ef, g\} = \{e f \partial(f)^{-1} \triangleright e^{-1}, (\partial(f^{-1}) \triangleright e) g (\partial(f^{-1}) \triangleright e^{-1})\} \{e, \partial(f) \triangleright g\}. \quad (15)$$

A very useful identity satisfied in any 2-crossed module is the following (here $x, a, e, a', e' \in E$):

$$\begin{aligned} \{x, e'^{-1} a'^{-1} e^{-1} a^{-1}\} &= ((xe'^{-1}) \triangleright' \{a'^{-1}, e^{-1}\}) \\ &\quad \{x, e'^{-1} (\partial(a')^{-1} \triangleright e^{-1}) a'^{-1} a^{-1}\} \partial(x) \triangleright (e'^{-1} \triangleright' \{a'^{-1}, e^{-1}\})^{-1}. \end{aligned} \quad (16)$$

By noting equation (6), using the fact that $(\delta: L \rightarrow E, \triangleright')$ is a crossed module, this is proved as:

$$\begin{aligned} &((xe'^{-1}) \triangleright' \{a'^{-1}, e^{-1}\}) \{x, e'^{-1} (\partial(a')^{-1} \triangleright e^{-1}) a'^{-1} a^{-1}\} \partial(x) \triangleright (e'^{-1} \triangleright' \{a'^{-1}, e^{-1}\})^{-1} \\ &= ((xe'^{-1}) \triangleright' \{a'^{-1}, e^{-1}\}) \{x, e'^{-1} (\partial(a')^{-1} \triangleright e^{-1}) a'^{-1} a^{-1}\} \\ &\quad (xe'^{-1}) \triangleright' (\{a'^{-1}, e^{-1}\}^{-1}) \{x, e'^{-1} \langle a'^{-1}, e^{-1} \rangle e'\} \\ &= (xe'^{-1} \langle a'^{-1}, e^{-1} \rangle e' x^{-1}) \triangleright' \{x, e'^{-1} (\partial(a')^{-1} \triangleright e^{-1}) a'^{-1} a^{-1}\} \{x, e'^{-1} \langle a'^{-1}, e^{-1} \rangle e'\} \\ &= \{x, e'^{-1} a'^{-1} e^{-1} a^{-1}\}. \end{aligned}$$

We have used equation (9), and the fact:

$$e'^{-1} a'^{-1} e^{-1} a^{-1} = e'^{-1} \langle a'^{-1}, e^{-1} \rangle e' e'^{-1} (\partial(a')^{-1} \triangleright e^{-1}) a'^{-1} a^{-1}.$$

We also have (here $x, e \in E$ and $k \in L$):

$$\{x, \delta(k)^{-1} e^{-1}\}^{-1} x \triangleright' k^{-1} = \partial(x) \triangleright k^{-1} \{x, e^{-1}\}^{-1}. \quad (17)$$

Which can be proved as (we use the fact that $(\delta: L \rightarrow E, \triangleright')$ is a crossed module):

$$\begin{aligned} \partial(x) \triangleright k^{-1} \{x, e^{-1}\}^{-1} &= \partial(x) \triangleright k^{-1} \{x, e^{-1}\}^{-1} (x \triangleright' k) (x \triangleright' k^{-1}) \\ &= (\partial(x) \triangleright k^{-1}) \{x, e^{-1}\}^{-1} (\partial(x) \triangleright' k) \{x, \delta(k)^{-1}\}^{-1} (x \triangleright' k^{-1}) \\ &= (\delta(\partial(x) \triangleright k^{-1}) \triangleright' \{x, e^{-1}\}^{-1}) \{x, \delta(k)^{-1}\}^{-1} (x \triangleright' k^{-1}) \\ &= \{x, \delta(k)^{-1} e^{-1}\}^{-1} x \triangleright' k^{-1}. \end{aligned}$$

Example 7 Given a pre-crossed module $E \rightarrow G$, consider the Peiffer subgroup $\langle E, E \rangle \subset E$, generated by the Peiffer commutators $\langle a, b \rangle$; see subsection 2.1. Then $\langle E, E \rangle \rightarrow E \rightarrow G$ is a 2-crossed module, where the Peiffer lifting is $\{a, b\} = \langle a, b \rangle$.

Example 8 (The underlying pre-crossed module functor T) There is a truncation functor T , or underlying pre-crossed module functor, sending a 2-crossed module $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ to its underlying pre-crossed module $(\partial: E \rightarrow G, \triangleright)$. This has a right adjoint sending a pre-crossed module $(\partial: E \rightarrow G, \triangleright)$ to $(\ker(\partial) \rightarrow E \rightarrow G, \triangleright)$, where we considered the inclusion map $\ker(\partial) \rightarrow E$, and the Peiffer lifting is as in the previous example. A left adjoint to the truncation functor was constructed in [21].

Example 9 (The underlying group Gr_0 and principal group Gr_1 functors) There is an underlying group functor Gr_0 sending a 2-crossed module $(L \rightarrow E \rightarrow G)$ to the group G . This has a right adjoint sending a group G to the 2-crossed module $(\{1\} \rightarrow G \rightarrow G)$, considering the identity map $G \rightarrow G$, the adjoint action of G on G ; and the trivial Peiffer lifting. Compare with example 2. On the other hand the principal group functor Gr_1 sends a 2-crossed module $(L \rightarrow E \rightarrow G)$ to the group E .

Definition 10 (Freeness up to order one) We say that 2-crossed module $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} F, \triangleright, \{, \})$ is free up to order one if $G = \text{Gr}_0(\mathcal{A})$ is a free group. An important category that we will consider is the category of free up to order one 2-crossed modules \mathcal{A} , with a specified (chosen) basis of $\text{Gr}_0(\mathcal{A})$.

Definition 11 (Homotopy groups of a 2-crossed module) Given a 2-crossed module $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ then both $\text{im}(\partial) \subset G$ and $\text{im}(\delta) \subset E$ are normal subgroups. This permits us to define the homotopy groups $\pi_i(\mathcal{A})$, where $i = 1, 2, 3$ as the first three homology groups of the underlying complex of \mathcal{A} .

2.3 The path space of a 2-crossed module

Let $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be a 2-crossed module. Let us define the path space $\mathcal{P}_*(\mathcal{G})$ of it, together with two surjective 2-crossed module morphisms $\mathcal{P}_*(\mathcal{G}) \xrightarrow[\text{Pr}_0^{\mathcal{G}}]{\text{Pr}_1^{\mathcal{G}}} \mathcal{G}$, and an inclusion $i_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{P}_*(\mathcal{G})$, with $\text{Pr}_1^{\mathcal{G}} \circ i_{\mathcal{G}} = \text{Pr}_0^{\mathcal{G}} \circ i_{\mathcal{G}} = \text{id}_{\mathcal{G}}$.

2.3.1 The derived action $*$ and the first and second lifted actions \bullet of a 2-crossed module

Most of this discussion appeared in [21].

Remark 12 (Convention on semidirect products) Given a left action \triangleright of the group G on the group E by automorphisms, the convention for the semidirect product $G \ltimes_{\triangleright} E$ is:

$$(g, e)(g', e') = (gg', (g'^{-1} \triangleright e)e').$$

In particular given $g \in G$ and $e \in E$ we have:

$$(g, e)^{-1} = (g^{-1}, g \triangleright e^{-1}).$$

Considering the inclusions $g \in G \mapsto (g, 1) \in G \ltimes_{\triangleright} E$ and $e \in E \mapsto (1, e) \in G \ltimes_{\triangleright} E$, then $ge = (g, e)$ and we have the commutation relation:

$$e g = g g^{-1} \triangleright e.$$

We resume the notation of 2.2. Consider the left action of E on L (the secondary action \triangleright' of \mathcal{G}) given by $e \triangleright' k \doteq k\{\delta(k)^{-1}, e\}$, where $e \in E$ and $k \in L$; lemma 5. Form the semidirect product $E \ltimes_{\triangleright'} L$. For the following see [15, 21].

Lemma 13 (Derived action) Let $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be a 2-crossed module. There exists a left action $*$ of E on $E \ltimes_{\triangleright'} L$, by automorphisms (called the “derived action of \mathcal{G} ”), with the form:

$$\begin{aligned} b * (e, k) &= (\partial(b) \triangleright e, (b \triangleright' \{b^{-1}, \partial(b) \triangleright e^{-1}\}) b \triangleright' k) \\ &= (\partial(b) \triangleright e, \{b, e^{-1}\}^{-1} b \triangleright' k), \text{ where } e, b \in E \text{ and } k \in L, \end{aligned}$$

see equation (12). Note that if $e \in E$ and $k \in L$:

$$b * (\delta(k), k^{-1}) = (\delta(\partial(b) \triangleright k), \partial(b) \triangleright k^{-1}). \quad (18)$$

Consider the group $E \rtimes_* (E \rtimes_{\triangleright'} L)$, whose group law is (remark 12):

$$\begin{aligned} (a, e, k)(a', e', k') &= \left(aa', (\partial(a'^{-1}) \triangleright e) e', \left((a' e')^{-1} \triangleright' (\{a', \partial(a')^{-1} \triangleright e^{-1}\} k) k' \right) \right) \\ &= \left(aa', (\partial(a'^{-1}) \triangleright e) e', \left(e'^{-1} \triangleright' (\{a'^{-1}, e^{-1}\}^{-1} a'^{-1} \triangleright' k) k' \right) \right). \end{aligned} \quad (19)$$

If we put $a = (a, 1, 1)$, $e = (1, e, 1)$ and $k = (1, 1, k)$, and the same for their images under \triangleright and \triangleright' we have:

$$(a, e, k) = aek, \quad aka^{-1} = a \triangleright' k, \quad eke^{-1} = e \triangleright' k. \quad (20)$$

Moreover, we have (21), (22) and (23), below:

$$aea^{-1} = (\partial(a) \triangleright e) (a \triangleright' \{a^{-1}, \partial(a) \triangleright e^{-1}\}) = (\partial(a) \triangleright e) (\{a, e^{-1}\})^{-1}, \quad (21)$$

$$kak^{-1} = a (a^{-1} \triangleright' k) k^{-1} \text{ and } eae^{-1} = a (\partial(a^{-1}) \triangleright e) (\{a^{-1}, e^{-1}\}^{-1}) e^{-1}, \quad (22)$$

$$klk^{-1} = \delta(k) \triangleright' l = (\delta(k), 1, 1) l (\delta(k), 1, 1)^{-1} = (1, \delta(k), 1) l (1, \delta(k), 1)^{-1}. \quad (23)$$

Particular cases of the multiplication are:

$$(a, 1, k)(a', 1, k') = (aa', 1, (a'^{-1} \triangleright k) k'), \text{ where } a, a' \in E \text{ and } k, k' \in L, \quad (24)$$

$$(1, e, k)(1, e', k') = (1, ee', (a'^{-1} \triangleright k) k'), \text{ where } e, e' \in E \text{ and } k, k' \in L. \quad (25)$$

Thus since $(\delta: L \rightarrow E, \triangleright')$ is a crossed module:

$$(\delta(l), 1, k)(\delta(l'), 1, k') = (\delta(l)\delta(l'), 1, l'^{-1}kl'k') \quad (26)$$

$$(1, \delta(l), k)(1, \delta(l'), k') = (1, \delta(l)\delta(l'), l'^{-1}kl'k'), \text{ where } k, k', l, l' \in L \quad (27)$$

Consider the semidirect product $G \rtimes_{\triangleright} E$, thus $(g, e)(g', e') = (gg', (g'^{-1} \triangleright e) e')$. The following essential lemma appeared in [21]. We will provide a proof since it is of fundamental importance for the sequel.

Lemma 14 (First lifted action) *Let $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be a 2-crossed module of groups. There exists a left action by automorphisms \bullet of $G \rtimes_{\triangleright} E$ on $E \rtimes_* (E \rtimes_{\triangleright'} L)$ (which we will call the first lifted action of \mathcal{G}), with the form:*

$$\begin{aligned} (g, x) \bullet (a, e, k) &= \left(g \triangleright a, g \triangleright ((\partial(a)^{-1} \triangleright x) e x^{-1}), g \triangleright \left((xe^{-1}) \triangleright' \{a^{-1}, x^{-1}\}^{-1} \right) g \triangleright \{x, e^{-1}a^{-1}\} (g\partial(x)) \triangleright k \right) \\ &\doteq g \triangleright \left(a, (\partial(a)^{-1} \triangleright x) e x^{-1}, (xe^{-1}) \triangleright' \{a^{-1}, x^{-1}\}^{-1} \{x, e^{-1}a^{-1}\} \partial(x) \triangleright k \right). \end{aligned}$$

Particular cases of the first lifted action, which will be important later are (it is instructive to prove that we do get an action by automorphisms in these cases, the following one being proved below).

$$x \bullet (a, e, k) = \left(a, (\partial(a)^{-1} \triangleright x) e x^{-1}, \left((xe^{-1}) \triangleright' \{a^{-1}, x^{-1}\}^{-1} \right) \{x, e^{-1}a^{-1}\} \partial(x) \triangleright k \right), \quad (28)$$

$$x \bullet (\delta(k), 1, l) = (\delta(k), 1, k^{-1} \partial(x) \triangleright (kl)), \quad \text{where } x \in E, \text{ and } k, l \in L, \quad (29)$$

$$x \bullet (1, e, k) = \left(a, xex^{-1}, \{x, e^{-1}\} \partial(x) \triangleright k \right), \quad (30)$$

$$(g, x) \bullet (1, e, k) = \left(1, g \triangleright (xex^{-1}), g \triangleright (\{x, e^{-1}\} \partial(x) \triangleright k) \right), \quad (31)$$

$$x \bullet (1, \delta(k), k^{-1}) = \left(1, x\delta(k)x^{-1}, \{x, \delta(k)^{-1}\} \partial(x) \triangleright k^{-1} \right) = \left(1, x\delta(k)x^{-1}, x \triangleright' k^{-1} \right), \quad (32)$$

$$x \bullet (a, 1, 1) = \left(a, \partial(a)^{-1} \triangleright x x^{-1}, (x \triangleright' \{a^{-1}, x^{-1}\}^{-1}) \{x, a^{-1}\} \right), \quad (33)$$

$$\delta(k) \bullet (a, 1, 1) = \left(a, \partial(a)^{-1} \triangleright \delta(k) \delta(k)^{-1}, k \partial(a) \triangleright k^{-1} \right), \quad (34)$$

and of course

$$g \bullet (a, e, k) = (g \triangleright a, g \triangleright e, g \triangleright k). \quad (35)$$

If x is such that $\{x, e\} = \{e, x\} = 1$, for all $e \in E$, we have:

$$x \bullet (a, e, k) = \left(a, (\partial(a)^{-1} \triangleright x) ex^{-1}, \partial(x) \triangleright k \right). \quad (36)$$

Proof. (Lemma 14) The only complicated bit is to prove that $E \subset G \ltimes_\triangleright E$ acts on $E \ltimes_* (E \ltimes_{\triangleright'} L)$ by automorphisms. Let us identify $x \in E$ with $(1, x, 1) \in E \ltimes_* (E \ltimes_{\triangleright'} L)$, and the same for its images under \triangleright . By the formulae above:

$$xae k x^{-1} = x a x^{-1} x e k x^{-1} = a(\partial(a)^{-1} \triangleright x) \{a^{-1}, x^{-1}\}^{-1} e k x^{-1} \quad (37)$$

$$= a(\partial(a)^{-1} \triangleright x) \{a^{-1}, x^{-1}\}^{-1} e x^{-1} (x \triangleright' k) \quad (38)$$

$$= a(\partial(a)^{-1} \triangleright x) e x^{-1} (x e^{-1} \triangleright' \{a^{-1}, x^{-1}\}^{-1}) (x \triangleright' k) \quad (39)$$

Therefore:

$$x \bullet (aek) = x(aek)x^{-1} f(x, aek) \quad (40)$$

where

$$f(x, aek) = (x \triangleright' k^{-1}) \left\{ x, e^{-1} a^{-1} \right\} \partial(x) \triangleright k.$$

To prove \bullet is an action, we need to prove that:

$$y f(x, aek) y^{-1} f\left(y, a(\partial(a)^{-1} \triangleright x) ex^{-1} \left(x e^{-1} \triangleright' \{a^{-1}, x^{-1}\}^{-1}\right) \{x, e^{-1} a^{-1}\} \partial(x) \triangleright k\right) = f(yx, aek). \quad (41)$$

The left hand side of (41) is:

$$\begin{aligned} & y(x \triangleright' k^{-1}) \left\{ x, e^{-1} a^{-1} \right\} (\partial(x) \triangleright k) y^{-1} \\ & y \triangleright' \left((\partial(x) \triangleright k^{-1}) \left\{ x, e^{-1} a^{-1} \right\}^{-1} (x e^{-1}) \triangleright' \{a^{-1}, x^{-1}\} \right) \{y, (a(\partial(a)^{-1} \triangleright x) ex^{-1})^{-1}\} \\ & \partial(y) \triangleright \left(((x e^{-1}) \triangleright' \{a^{-1}, x^{-1}\}^{-1}) \{x, e^{-1} a^{-1}\} \partial(x) \triangleright k \right), \end{aligned} \quad (42)$$

which simplifies to (on the nose, by using (20)):

$$\begin{aligned} & ((yx) \triangleright' k^{-1}) (y x e^{-1}) \triangleright' \{a^{-1}, x^{-1}\} \{y, (a(\partial(a)^{-1} \triangleright x) ex^{-1})^{-1}\} \\ & \partial(y) \triangleright \left((x e^{-1}) \triangleright' \{a^{-1}, x^{-1}\}^{-1} \right) \partial(y) \triangleright \{x, e^{-1} a^{-1}\} \partial(yx) \triangleright k. \end{aligned} \quad (43)$$

The right hand side of (41) is:

$$((yx) \triangleright' k^{-1}) \left\{ yx, e^{-1} a^{-1} \right\} \partial(yx) \triangleright k = ((yx) \triangleright' k^{-1}) \{y, x e^{-1} a^{-1} x^{-1}\} \partial(y) \triangleright \{x, e^{-1} a^{-1}\} \partial(yx) \triangleright k. \quad (44)$$

To prove (41) we thus need to prove:

$$\{y, x e^{-1} a^{-1} x^{-1}\} = (y x e^{-1}) \triangleright' \{a^{-1}, x^{-1}\} \{y, (a(\partial(a)^{-1} \triangleright x) ex^{-1})^{-1}\} \partial(y) \triangleright \left((x e^{-1}) \triangleright' \{a^{-1}, x^{-1}\}^{-1} \right).$$

This follows from equation (16), with $x = y$, $e'^{-1} = x e^{-1}$, $a'^{-1} = a^{-1}$, $e^{-1} = x^{-1}$ and $a = 1$.

To prove that the action of $x \in E$ on $E \ltimes_* (E \ltimes_{\triangleright'} L)$ is by automorphisms we need to prove that

$$(x, aek a' e' k') = (x \triangleright' k'^{-1}) ((x a' e' x^{-1})^{-1} \triangleright' f(x, aek)) (x \triangleright' k') f(x, a' e' k'). \quad (45)$$

The left hand side of (45) is, by (19):

$$\begin{aligned} & f\left(x, aa'(\partial(a'^{-1}) \triangleright e) e' \left((a' e')^{-1} \triangleright' (\{a', \partial(a')^{-1} \triangleright e^{-1}\} k) k' \right) \right) = \\ & x' \triangleright' (k'^{-1}) x \triangleright' \left((a' e')^{-1} \triangleright' (\{a', \partial(a')^{-1} \triangleright e^{-1}\} k) \right)^{-1} \{x, (aa'(\partial(a'^{-1}) \triangleright e) e')^{-1}\} \\ & \partial(x) \triangleright \left(\left((a' e')^{-1} \triangleright' (\{a', \partial(a')^{-1} \triangleright e^{-1}\} k) \right) k' \right). \end{aligned} \quad (46)$$

The right hand side of (45) is:

$$(x \triangleright' k'^{-1}) (xa'e'x^{-1})^{-1} \triangleright' \left((x \triangleright' k^{-1}) \{x, e^{-1}a^{-1}\} \partial(x) \triangleright k \right) \{x, e'^{-1}a'^{-1}\} \partial(x) \triangleright k'. \quad (47)$$

To prove (45) we need to prove:

$$\begin{aligned} x \triangleright' \left((a'e')^{-1} \triangleright' (\{a', \partial(a')^{-1} \triangleright e^{-1}\})^{-1} \right) \{x, (aa'(\partial(a'^{-1}) \triangleright e)e')^{-1}\} \partial(x) \triangleright \left((a'e')^{-1} \triangleright' (\{a', \partial(a')^{-1} \triangleright e^{-1}\}k) \right) \\ = (xa'e'x^{-1})^{-1} \triangleright' \left(\{x, e^{-1}a^{-1}\} \partial(x) \triangleright k \right) \{x, e'^{-1}a'^{-1}\}, \end{aligned} \quad (48)$$

where the bottom term simplifies as (since $(\delta: L \rightarrow E, \triangleright')$ is a crossed module):

$$\begin{aligned} (xa'e'x^{-1})^{-1} \triangleright' \left(\{x, e^{-1}a^{-1}\} \partial(x) \triangleright k \right) \{x, e'^{-1}a'^{-1}\} \\ = (xa'e'x^{-1})^{-1} \triangleright' \{x, e^{-1}a^{-1}\} (xa'e'x^{-1})^{-1} \triangleright' (\partial(x) \triangleright k) \{x, e'^{-1}a'^{-1}\} \\ = (xa'e'x^{-1})^{-1} \triangleright' \{x, e^{-1}a^{-1}\} \{x, e'^{-1}a'^{-1}\} (\langle x, e'^{-1}a'^{-1} \rangle^{-1} (xa'e'x^{-1})^{-1}) \triangleright' (\partial(x) \triangleright k) \\ = (xa'e'x^{-1})^{-1} \triangleright' \{x, e^{-1}a^{-1}\} \{x, e'^{-1}a'^{-1}\} \partial(x) \triangleright \left((a'e')^{-1} \triangleright' k \right) = \{x, e'^{-1}a'^{-1}e^{-1}a^{-1}\} \partial(x) \triangleright ((a'e')^{-1} \triangleright' k). \end{aligned}$$

And the top term simplifies, as by (12):

$$\left((xe'^{-1}) \triangleright' (\{a'^{-1}, e^{-1}\}) \right) \{x, (aa'(\partial(a'^{-1}) \triangleright e)e')^{-1}\} \partial(x) \triangleright \left(e'^{-1} \triangleright' \{a'^{-1}, e^{-1}\}^{-1} \right) \partial(x) \triangleright ((a'e')^{-1} \triangleright' k). \quad (49)$$

Equation (48), thus equation (45), follows therefore plainly from (16). ■

It easily follows that, where ad is the adjoint action of L on itself:

Lemma 15 (Second lifted action) *There is a left action \bullet of $G \ltimes_{\triangleright} E$ in $L \ltimes_{\text{ad}} L$, by automorphisms (called the second lifted action), which has the form*

$$x \bullet (k, l) = (k, k^{-1} \partial(x) \triangleright (kl)) \text{ and } g \bullet (k, l) = (g \triangleright k, g \triangleright l).$$

Here $x \in E$, $g \in G$ and $(k, l) \in L \ltimes_{\text{ad}} L$.

2.3.2 Definition of the path space $\mathcal{P}_*(\mathcal{G})$ of a 2-crossed module \mathcal{G}

By using (29), (26) and (19), we can see that the maps

$$(a, e, k) \in E \ltimes_* (E \ltimes_{\triangleright'} L) \xrightarrow{\beta} (\partial(a), e) \in G \ltimes_{\triangleright} E$$

and

$$(k, l) \in L \ltimes_{\text{ad}} L \xrightarrow{\alpha} (\delta(k), 1_E, l) \in E \ltimes_* (E \ltimes_{\triangleright'} L)$$

are $G \ltimes_{\triangleright} E$ -equivariant group morphisms, with respect to the lifted actions \bullet of \mathcal{G} and the adjoint action of $G \ltimes_{\triangleright} E$ on itself. This defines a chain complex of groups:

$$L \ltimes_{\text{ad}} L \xrightarrow{\alpha} E \ltimes_* (E \ltimes_{\triangleright'} L) \xrightarrow{\beta} G \ltimes_{\triangleright} E. \quad (50)$$

The Peiffer pairing in the pre-crossed module $E \ltimes_* (E \ltimes_{\triangleright'} L) \xrightarrow{\beta} G \ltimes_{\triangleright} E$ was calculated in [21]:

$$\langle (a, e, k), (a', e', k') \rangle = (\langle a, a' \rangle, 1, \{a, a'\}^{-1} \{ae\delta(k), a'e'\delta(k')\}). \quad (51)$$

By using example 7, and the structure of the group complex (50), namely equations (26) and the form of the product in $L \ltimes_{\text{ad}} L$, there follows that there exists a 2-crossed module structure in (50), whose Peiffer lifting $|\cdot|: (E \ltimes_* (E \ltimes_{\triangleright'} L)) \times (E \ltimes_* (E \ltimes_{\triangleright'} L)) \rightarrow L \ltimes_{\text{ad}} L$ takes the following form:

$$|(a, e, k), (a', e', k')| = (\{a, a'\}, \{a, a'\}^{-1} \{ae\delta(k), a'e'\delta(k')\}). \quad (52)$$

Thus on generators we have:

$$|(a, 1, 1), (a', 1, 1)| = (\{a, a'\}, 1), \quad |(a, 1, 1), (1, e', 1)| = (1, \{a, e'\}), \quad (53)$$

$$|(1, e, 1), (a', 1, 1)| = (1, \{e, a'\}), \quad |(a, 1, 1), (1, 1, k')| = (1, \{a, \delta(k')\}), \quad (54)$$

$$|(1, e, 1), (1, e', 1)| = (1, \{e, e'\}), \quad |(1, e, 1), (1, 1, k')| = (1, \{e, \delta(k')\}), \quad (55)$$

$$|(1, 1, k), (a', 1, 1)| = (1, \{\delta(k), a'\}), \quad |(1, 1, k), (1, e', 1)| = (1, \{\delta(k), e'\}), \quad (56)$$

$$|(1, 1, k), (1, 1, k')| = (1, [k, k']). \quad (57)$$

Definition 16 For a 2-crossed module $(L \rightarrow E \rightarrow G, \triangleright, \{ \})$, the 2-crossed module

$$\mathcal{P}_*(\mathcal{G}) = \left(L \ltimes_{\text{ad}} L \xrightarrow{\alpha} E \ltimes_* (E \ltimes_{\triangleright'} L) \xrightarrow{\beta} G \ltimes_{\triangleright} E, \bullet, |, | \right) \quad (58)$$

just defined will be called the (pointed) path-space of \mathcal{G} . Clearly the path-space construction \mathcal{P}_* is functorial with respect to 2-crossed module morphisms.

For $a, e \in E$ and $k \in L$, put $a = (a, 1, 1)$, $e = (1, e, 1)$ and $k = (1, 1, k)$. Given $m, l \in L$ we have (where \bullet' denotes the secondary action of $\mathcal{P}_*(\mathcal{G})$, definition 5, an action by automorphisms of $E \ltimes_* (E \ltimes_{\triangleright'} L)$ on $L \ltimes_{\text{ad}} L$):

$$\begin{aligned} a \bullet' (m, 1) &= (a \triangleright' m, 1), & a \bullet' (1, l) &= (1, a \triangleright' l) \\ e \bullet' (m, 1) &= (m, \{\delta(m^{-1}), e\}), & e \bullet' (1, l) &= (1, e \triangleright' l) \\ k \bullet' (m, 1) &= (m, m^{-1} k m k^{-1}). & k \bullet' (1, l) &= (1, k l k^{-1}) = (1, \delta(k) \triangleright' l). \end{aligned} \quad (59)$$

By straightforward calculations we conclude (to this end note $(a, e, k) = aek$) that:

Theorem 17 Let $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{ \})$ be a 2-crossed module. The maps $(k, l) \in L \ltimes_{\text{ad}} L \xrightarrow{p'} kl \in L$ and

$$(a, e, k) \in E \ltimes_* (E \ltimes_{\triangleright'} L) \xrightarrow{q'} ae\delta(k) \in E, \quad (g, e) \in G \ltimes E \xrightarrow{r'} g\partial(e) \in G;$$

and also:

$$(k, l) \in L \ltimes_{\text{ad}} L \xrightarrow{p} k \in L, \quad (a, e, k) \in E \ltimes_* (E \ltimes_{\triangleright'} L) \xrightarrow{q} a \in E, \quad (g, e) \in G \ltimes E \xrightarrow{r} g \in G$$

are group morphisms. Moreover the triples $\text{Pr}_0^{\mathcal{G}} \doteq (p, q, r)$ and $\text{Pr}_1^{\mathcal{G}} \doteq (p', q', r')$ define surjective morphisms $\mathcal{P}_*(\mathcal{G}) \rightarrow \mathcal{G}$ of 2-crossed modules. We also have an inclusion map $i_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{P}_*(\mathcal{G})$ such that:

$$g \mapsto (g, 1), \quad e \mapsto (e, 1, 1), \quad k \mapsto (k, 1).$$

Therefore $\text{Pr}_0^{\mathcal{G}} \circ i_{\mathcal{G}}$ and $\text{Pr}_1^{\mathcal{G}} \circ i_{\mathcal{G}}$ each are the identity of \mathcal{G} . Moreover the map $(\text{Pr}_0^{\mathcal{G}}, \text{Pr}_1^{\mathcal{G}}): \mathcal{P}_*(\mathcal{G}) \rightarrow \mathcal{G} \times \mathcal{G}$ is a fibration of 2-crossed modules, considering the model category structure in the category of 2-crossed modules defined in [13], see the introduction. This is because both maps $(p, p'): L \ltimes_{\text{ad}} L \rightarrow L \times L$ and $(q, q'): E \ltimes_* (E \ltimes_{\triangleright'} L) \rightarrow E \times E$ are surjective. Therefore $\mathcal{P}_*(\mathcal{G})$ is a good path-space for \mathcal{G} , [18], since clearly $i_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{P}_*(\mathcal{G})$ induces isomorphism at the level of homotopy groups.

2.3.3 Some particular group morphisms

We note some particular group morphisms which will be used several times later. Note that if $(\partial: E \rightarrow G, \triangleright)$ is a pre-crossed module then the map:

$$(g, e) \in G \ltimes_{\triangleright} E \mapsto g\partial(e) \in G \quad (60)$$

is a group morphism, co-variant with respect to the actions \triangleright of G (by automorphisms), where $g \triangleright (g', e) = (gg'g^{-1}, g \triangleright e)$ and $g \triangleright h = ghg^{-1}$. Moreover, if $(\psi, \phi): (\partial: E \rightarrow G, \triangleright) \rightarrow (\partial': E' \rightarrow G', \triangleright')$ is a pre-crossed module map, then:

$$(g, e) \in G \ltimes_{\triangleright} E \mapsto (\phi(g), \psi(e)) \in G' \ltimes_{\triangleright'} E'. \quad (61)$$

is also a group morphism, and of course so is the map

$$(g, e) \in G \ltimes_{\triangleright} E \mapsto \phi(g) \partial(\psi(e)) \in G'. \quad (62)$$

Combining with the construction of the first lifted action, this gives a several group morphisms. For example

$$(g, x, a, e, k) \in (G \ltimes_{\triangleright} E) \triangleright_{\bullet} (E \ltimes_* (E \ltimes_{\triangleright} L)) \xrightarrow{\nu} (g\partial(x), ae\delta(k)) \in G \ltimes_{\triangleright} E \quad (63)$$

(given by the pair (g', r') in theorem 17). Also, given the pre-crossed module structure of the path-space crossed module we have a group morphism

$$(g, x, a, e, k) \in (G \ltimes_{\triangleright} E) \triangleright_{\bullet} (E \ltimes_* (E \ltimes_{\triangleright} L)) \mapsto (g, x) \beta(a, e, k) = (g\partial(a), \partial(a)^{-1} \triangleright x e) \in G \ltimes_{\triangleright} E \quad (64)$$

which is invariant under the action \bullet of $G \ltimes_{\triangleright} E$, such that:

$$(h, y) \bullet (g, x, a, e, k) = ((h, y)(g, x)(h, y)^{-1}, (h, y) \bullet (a, e, k))$$

and

$$(h, y) \bullet (g, x) \doteq (h, y)(g, x)(h, y)^{-1}.$$

2.4 The disk space of a 2-crossed module

2.4.1 The double path-space of a 2-crossed module

Given a 2-crossed module $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$, we can iterate the path-space construction, obtaining a 2-crossed module $\mathcal{P}_*(\mathcal{P}_*(\mathcal{G}))$, the double path-space of \mathcal{G} , with underlying group complex:

$$\begin{aligned} (L \ltimes_{\text{ad}} L) \ltimes_{\text{ad}} (L \ltimes_{\text{ad}} L) &\xrightarrow{\alpha^{\sharp}} (E \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_* ((E \ltimes_* (E \ltimes_{\triangleright'} L) \ltimes_{\bullet'} (L \ltimes_{\text{ad}} L)) \\ &\xrightarrow{\beta^{\sharp}} (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (E \ltimes_* (E \ltimes_{\triangleright'} L)), \end{aligned} \quad (65)$$

and lifted actions (now denoted by \square) of the first group on the remaining groups. Here \bullet' denotes the secondary action of $\mathcal{P}_*(\mathcal{G})$; lemma 5.

There are four natural 2-crossed module maps $\mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) \rightarrow \mathcal{P}_*(\mathcal{G})$. Namely the maps $\text{Pr}_1^{\mathcal{P}_*(\mathcal{G})}$ and $\text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}$ of theorem 17, which have the form (respectively):

$$\begin{aligned} (k, l, k', l') &\mapsto (kk', k'^{-1}lk'l'), & (a, e, k, a', e', k', l, l') &\mapsto (a, e, k)(a', e', k')(\delta(l), 1, l'), \\ (g, x, a, e, k) &\mapsto (g, x)(\partial(a), e) = (g\partial(a), \partial(a)^{-1} \triangleright x e) \end{aligned}$$

and

$$(k, l, k', l') \mapsto (k, l), \quad (a, e, k, a', e', k', l, l') \mapsto (a, e, k), \quad (g, x, a, e, k) \mapsto (g, x).$$

Also, by applying the path-space functor \mathcal{P}_* to the 2-crossed module maps $\text{Pr}_1^{\mathcal{G}}$ and $\text{Pr}_0^{\mathcal{G}}$, from $\mathcal{P}_*(\mathcal{G})$ to \mathcal{G} , yields 2-crossed module maps $\mathcal{P}_*(\text{Pr}_1^{\mathcal{G}}), \mathcal{P}_*(\text{Pr}_0^{\mathcal{G}}): \mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) \rightarrow \mathcal{P}_*(\mathcal{G})$, which have the form, respectively:

$$\begin{aligned} (k, l, k', l') &\mapsto (kl, k'l'), & (a, e, k, a', e', k', l, l') &\mapsto (ae\delta(k), a'e'\delta(k'), ll'), & (g, x, a, e, k) &\mapsto (g\partial(x), ae\delta(k)), \\ (k, l, k', l') &\mapsto (k, k'), & (a, e, k, a', e', k', l, l') &\mapsto (a, a', l), & (g, x, a, e, k) &\mapsto (g, a). \end{aligned}$$

2.4.2 Definition of the disk space $\mathcal{D}_*(\mathcal{G})$ of a 2-crossed module \mathcal{G}

Let $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be a 2-crossed module. Its disk space $\mathcal{D}_*(\mathcal{G})$ will be constructed as being an embedded 2-crossed module, within the double path space $\mathcal{P}_*(\mathcal{P}_*(\mathcal{G}))$ of \mathcal{G} . Namely, in the last bit of (65), instead of $E \ltimes_* (E \ltimes_{\triangleright'} L)$, we put the subgroup isomorphic to L , of elements of the form $(1, \delta(k), k^{-1})$, where $k \in L$. The group law is

$$(1, \delta(k), k^{-1})(1, \delta(l), l^{-1}) = (1, \delta(kl), (kl)^{-1});$$

see (25). Under the identification $k = (1, \delta(k), k^{-1})$, the restriction of the first lifted action \bullet of \mathcal{G} to this subgroup is, by (32):

$$(g, x) \bullet k = g \triangleright (x \triangleright' k). \quad (66)$$

As far as the second group of the disk $\mathcal{D}_*(\mathcal{G})$ is concerned, we consider the subgroup of $(E \rtimes_* (E \rtimes_{\triangleright'} L)) \rtimes_{\bullet} (L \rtimes_{\text{ad}} L)$, isomorphic to L , of elements of the form:

$$(1, \delta(k), k^{-1}, 1, 1).$$

By equations (52) and (3), we have, where $|\cdot|$ is the Peiffer lifting in $\mathcal{P}_*(\mathcal{G})$:

$$|(a, e, k), (1, \delta(l), l^{-1})| = (1_L, 1_L) = |(1, \delta(l), l^{-1}), (a, e, k)|.$$

This will be used several times in the following calculations. The restriction of the derived action $*$ of $E \rtimes_* (E \rtimes_{\triangleright'} L)$ on $(E \rtimes_* (E \rtimes_{\triangleright'} L)) \rtimes_{\bullet} (L \rtimes_{\text{ad}} L)$ to this isomorphic image of L is, given by

$$(a, e, l) * (1, \delta(k), k^{-1}, 1, 1) = (\beta(a, e, l) \bullet (1, \delta(k), k^{-1}), 1, 1) = (1, \partial(a) \triangleright (e\delta(k)e^{-1}), \partial(a) \triangleright (e \triangleright' k^{-1}), 1, 1).$$

Thus under the identification $k = (1, \delta(k), k^{-1}, 1, 1)$ we have

$$(a, e, l) * k = \partial(a) \triangleright (e \triangleright' k).$$

We now describe the restriction of the lifted action \square of $\mathcal{P}_*(\mathcal{G})$. By (36):

$$\begin{aligned} (1, \delta(k), k^{-1}) \square (a, e, l, 1, \delta(m), m^{-1}, 1, 1) \\ = (a, e, l, 1, e^{-1} (\partial(a)^{-1} \triangleright \delta(k)) e \delta(m) \delta(k)^{-1}, km^{-1}e^{-1} \triangleright' (\partial(a)^{-1} \triangleright k^{-1}), 1, 1). \end{aligned} \quad (67)$$

Denoting the Peiffer lifting in the double path space and the path space by $\{\cdot, \cdot\}$ and $|\cdot, \cdot|$, respectively, we have, by (52):

$$\{(a, e, k, 1, \delta(l), l^{-1}, 1, 1), (a', e', k', 1, \delta(l'), l'^{-1}, 1, 1)\} = (|aek, a'e'k'|, |aek, a'e'k'|^{-1} |aek\delta(l)l^{-1}, a'e'k'\delta(l')l'^{-1}|)$$

Now note that, by (25), where $\{\cdot, \cdot\}$ is the Peiffer lifting in \mathcal{G} :

$$\begin{aligned} |aek\delta(l)l^{-1}, a'e'k'\delta(l')l'^{-1}| &= |ae\delta(l) (\delta(l^{-1}) \triangleright' k) l^{-1}, a'e'\delta(l') (\delta(l')^{-1} \triangleright' k') l'^{-1}| \\ &= (\{a, a'\}, \{a, a'\}^{-1} \{ae\delta(l)\delta(\delta(l^{-1}) \triangleright' kl^{-1}), a'e'\delta(l')\delta(\delta(l')^{-1} \triangleright' k'l'^{-1})\}) \\ &= (\{a, a'\}, \{a, a'\}^{-1} \{ae\delta(k), a'e'\delta(k')\}) \\ &= |aek, a'e'k'|. \end{aligned}$$

Therefore

$$\{(a, e, k, 1, \delta(l), l^{-1}, 1, 1), (a', e', k', 1, \delta(l'), l'^{-1}, 1, 1)\} = (|aek, a'e'k'|, 1, 1)$$

We thus have the following theorem:

Theorem 18 *Given a 2-crossed module $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{\cdot, \cdot\})$, there exists a 2-crossed module $\mathcal{D}_*(\mathcal{G})$, called the disk-space of \mathcal{G} , with underlying complex of groups:*

$$L \rtimes_{\text{ad}} L \xrightarrow{\alpha^2} (E \rtimes_* (E \rtimes_{\triangleright'} L)) \rtimes_* L \xrightarrow{\beta^2} (G \rtimes_{\triangleright} E) \rtimes_{\bullet} L,$$

where (recall remark 12):

$$(g, e) \bullet k = g \triangleright (e \triangleright' k) \quad (a, e, k) * l = \partial(a) \triangleright (e \triangleright' l). \quad (68)$$

The underlying action (denoted by \square) of $(G \rtimes_{\triangleright} E) \rtimes_{\bullet} L$ on $(E \rtimes_* (E \rtimes_{\triangleright'} L)) \rtimes_* L$ is given by (where $(g, e) \in G \rtimes E$):

$$(g, e) \square (a, f, l, l') = ((g, e) \bullet (a, f, l), g \triangleright (e \triangleright' l')) \quad (69)$$

and by (where $k \in L$):

$$k \square (a, e, l, l') = (a, e, l, e^{-1} \triangleright' (\partial(a)^{-1} \triangleright k) l' k^{-1}). \quad (70)$$

The action \square of $(G \rtimes_{\triangleright} E) \rtimes_{\bullet} L$ on $L \rtimes_{\text{ad}} L$ has the form:

$$(g, e, l) \square (k, k') = (g, e) \bullet (k, k'). \quad (71)$$

The boundary maps are:

$$\alpha^2(k, l) = (\delta(k), 1, l, 1) \text{ and } \beta^2(a, e, k, l) = (\partial(a), e, l).$$

And finally the Peiffer lifting is (recall (52)):

$$\begin{aligned} \{(a, e, k, l), ((a, e, k, l))\} &= |(a, e, k), (a', e', k')| \\ &= \{a, a'\}, \{a, a'\}^{-1} \{ae\delta(k), a'e'\delta(k')\}. \end{aligned} \quad (72)$$

The disk space $\mathcal{D}_*(\mathcal{G})$ has an inclusion map into $\mathcal{P}_*(\mathcal{P}_*(\mathcal{G}))$ of the form:

$$\begin{aligned} (g, x, k) &\mapsto (g, x, 1, \delta(k), k^{-1}), \\ (a, e, k, l) &\mapsto (a, e, k, 1, \delta(l), l^{-1}, 1, 1), \\ (k, l) &\mapsto (k, l, 1, 1). \end{aligned}$$

Moreover the maps $(p, q, r): \mathcal{D}_*(\mathcal{G}) \rightarrow \mathcal{P}_*(\mathcal{G})$ and $(p', q', r'): \mathcal{D}_*(\mathcal{G}) \rightarrow \mathcal{P}_*(\mathcal{G})$, where:

$$p(k, l) = (k, l), \quad q(a, e, k, l) = (a, e, k), \quad r(g, e, k) = (g, e), \quad (73)$$

$$p'(k, l) = (k, l), \quad q(a, e, k, l) = (a, e\delta(l), l^{-1}k), \quad r'(g, e, k) = (g, e\delta(k)), \quad (74)$$

are morphisms of 2-crossed modules, and are obtained from the composition of the inclusion map $\mathcal{D}_*(\mathcal{G}) \rightarrow \mathcal{P}_*(\mathcal{P}_*(\mathcal{G}))$ and the projection maps $\text{Pr}_1^{\mathcal{P}_*(\mathcal{G})}, \text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}: \mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) \rightarrow \mathcal{P}_*(\mathcal{G})$. Note:

$$(a, e, k)(1, \delta(l), l^{-1}) = (a, e\delta(l), (\delta(l)^{-1} \triangleright k) l^{-1}) = (a, e\delta(l), l^{-1}k).$$

Remark 19 A coordinate free definition of the disk space $\mathcal{D}_*(\mathcal{G})$ of a 2-crossed module \mathcal{G} is as the limit of:

$$\begin{array}{ccccc} & \mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) & & \mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) & \\ & \downarrow \mathcal{P}_*(\text{Pr}_1^{\mathcal{G}}) & \searrow \text{id} & \swarrow \text{id} & \downarrow \mathcal{P}_*(\text{Pr}_0^{\mathcal{G}}) \\ \mathcal{G} & \xrightarrow{i_{\mathcal{G}}} \mathcal{P}_*(\mathcal{G}) & & \mathcal{P}_*(\mathcal{G}) & \xleftarrow{i_{\mathcal{G}}} \mathcal{G} \end{array} \quad (75)$$

This is the point of view considered in [25].

2.5 The composer pre-crossed module

Let $\mathcal{G} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be a 2-crossed module. The composer pre-crossed module $\text{Comp}(\mathcal{G})$ of \mathcal{G} , which can intuitively be seen as underlying pre-crossed module of the triangle space of \mathcal{G} , will (not surprisingly) have a primary importance in the construction of the concatenation of 2-crossed module homotopies.

2.5.1 Definition of the composer pre-crossed module $\text{Comp}(\mathcal{G})$ of a crossed module \mathcal{G}

Let us look at the underlying pre-crossed module of the double path-space 2-crossed module $\mathcal{P}_*(\mathcal{P}_*(\mathcal{G}))$; equation (65). In addition to the underlying pre-crossed module of the disk space of \mathcal{G} , there exists another naturally embedded pre-crossed module $\text{Comp}(\mathcal{G})$, strictly containing the underlying pre-crossed module of the disk space of \mathcal{G} . In the last bit of (65), instead of $(G \ltimes_{\triangleright} E) \ltimes_{\bullet} (E \ltimes_{*} (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\mathcal{P}_*(\mathcal{P}_*(\mathcal{G})))$, example 9, let us consider:

$$(G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L)) \doteq \text{Gr}_0(\text{Comp}(\mathcal{G})),$$

where the restriction of the \bullet -action is as in (31), namely

$$(g, x) \bullet (1, e, k) = \left(1, g \triangleright (xex^{-1}), g \triangleright (\{x, e^{-1}\} \partial(x) \triangleright k)\right). \quad (76)$$

In particular:

$$(g, x) \bullet (1, \delta(k), k^{-1}) = \left(1, g \triangleright (x\delta(k)x^{-1}), g \triangleright (\{x, \delta(k)^{-1}\} \partial(x) \triangleright k^{-1})\right) = \left(1, g \triangleright (x\delta(k)x^{-1}), g \triangleright (x\delta(k)x^{-1})\right), \quad (77)$$

where we have used (6).

Looking again at the second component of the 2-crossed module (65), we now consider the subgroup of $(E \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_* ((E \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_{\bullet'} (L \ltimes_{\text{ad}} L)) \doteq \text{Gr}_1(\mathcal{P}_*(\mathcal{P}_*(\mathcal{G})))$, example 9, of the form

$$(E \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L)) \doteq \text{Gr}_1(\text{Comp}(\mathcal{G})).$$

Note that (where \bullet' denotes the secondary action of $\mathcal{P}_*(\mathcal{G})$, lemma 5, and $|\cdot|$ the Peiffer lifting in $\mathcal{P}_*(\mathcal{G})$):

$$(a, e, k) \bullet' (1, l) = (1, l) |(1, 1, \delta(l)^{-1}), (a, e, k)| = (1, l)(1, \{\delta(l)^{-1}, ae\delta(k)\} = (1, (ae\delta(k)) \triangleright' l) = (1, (ae) \triangleright' (klk^{-1})).$$

We have used (59). Also, considering the derived action of $\mathcal{P}_*(\mathcal{G})$; lemma 13:

$$\begin{aligned} (a, e, k) * ((1, f, l), (1, m)) &= ((\partial(a), e) \bullet (1, f, l), |(a, e, k), (1, f^{-1}, f \triangleright' l^{-1})|^{-1} (a, e, k) \bullet' (1, m)) \\ &= ((\partial(a), e) \bullet (1, f, l), (1, \{ae\delta(k), \delta(l^{-1})f^{-1}\}^{-1}(1, (ae) \triangleright' (kmk^{-1}))). \end{aligned}$$

Thus the product in $(E \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L)) = \text{Gr}_1(\text{Comp}(\mathcal{G}))$ is:

$$\begin{aligned} ((a, e, k), (1, f, l), (1, m)) &((a', e', k'), (1, f', l'), (1, m')) \\ &= \left((a, e, k)(a', e', k'), ((\partial(a'), e')^{-1} \bullet (1, f, l) (1, f', l')), \right. \\ &\quad \left. (1, (f' \delta(l'))^{-1} \triangleright' \{(a' e' \delta(k'))^{-1}, (f \delta(l))^{-1}\}^{-1} (1, (f' \delta(l'))^{-1} \triangleright' (a' e' \delta(k'))^{-1} \triangleright' m) (1, m')) \right). \end{aligned} \quad (78)$$

The restriction Δ of the lifted action of $\mathcal{P}_*(\mathcal{G})$ to an action by automorphisms of $\text{Gr}_0(\text{Comp}(\mathcal{G}))$ on $\text{Gr}_1(\text{Comp}(\mathcal{G}))$ is:

$$\begin{aligned} (g, x, 1, z, w) \Delta ((a, e, k), (1, f, l), (1, m)) &= (g, x) \bullet ((a, e, k), (\partial(a), e)^{-1} \bullet (1, z, w) (1, f, l) (1, z, w)^{-1}, \\ &\quad ((1, z, w)(1, f, l)^{-1}) \bullet' (1, \{\delta(k)^{-1}e^{-1}a^{-1}, \delta(w)^{-1}z^{-1}\}^{-1}(1, \{z\delta(w), (ae\delta(k)f\delta(l))^{-1}\}(1, \partial(z) \triangleright m)) \\ &= (g, x) \bullet \left((a, e, k), (\partial(a), e)^{-1} \bullet (1, z, w) (1, f, l) (1, z, w)^{-1}, \right. \\ &\quad \left. (1, (z\delta(wl^{-1})f^{-1}) \triangleright' (\{(ae\delta(k))^{-1}, \delta(w)^{-1}z^{-1}\}^{-1} \{z\delta(w), (ae\delta(k)f\delta(l))^{-1}\} \partial(z) \triangleright m)) \right) \end{aligned} \quad (79)$$

To complete the construction of the pre-crossed module $\text{Comp}(\mathcal{G})$, note that the boundary map

$$\text{Gr}_1(\text{Comp}(\mathcal{G})) = (E \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L)) \xrightarrow{\beta'} (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{G}))$$

is:

$$\beta'((a, e, k), (1, f, l), (1, m)) = (\partial(a), e, 1, f, l). \quad (80)$$

The restrictions of the projection maps $\text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}, \text{Pr}_1^{\mathcal{P}_*(\mathcal{G})}: \mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) \rightarrow \mathcal{P}_*(\mathcal{G})$ to $\text{Comp}(\mathcal{G})$ are (respectively):

$$q(a, e, k, 1, f, l, 1, m) = (a, e, k), \quad r(g, x, 1, e, k) = (g, x), \quad (81)$$

$$q'(a, e, k, 1, f, l, 1, m) = (a, ef, (f^{-1} \triangleright' k)lm), \quad r'(g, x, 1, e, k) = (g, xe). \quad (82)$$

On the other hand the restrictions of $\mathcal{P}_*(\text{Pr}_1^{\mathcal{G}})$ and $\mathcal{P}_*(\text{Pr}_0^{\mathcal{G}})$ to $\text{Comp}(\mathcal{G})$ and are given by, respectively:

$$u(a, e, k, 1, f, l, 1, m) = (ae\delta(k), f\delta(l), m), \quad v(g, x, 1, e, k) = (g\partial(x), e\delta(k)), \quad (83)$$

$$u'(a, e, k, 1, f, l, 1, m) = (a, 1, 1), \quad v'(g, x, 1, e, k) = (g, 1). \quad (84)$$

Remark 20 A succinct construction of the composer pre-crossed module is by defining it as being the underlying pre-crossed module of the 2-crossed module given by the limit of the pull-back diagram:

$$\begin{array}{ccc} & \mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) & (85) \\ & \downarrow \mathcal{P}_*(\text{Pr}_0^{\mathcal{G}}) & \\ \mathcal{G} & \xrightarrow{i_{\mathcal{G}}} \mathcal{P}_*(\mathcal{G}) & \end{array}$$

2.5.2 Some more particular group morphisms

Note that, in $E \ltimes_* (E \ltimes_{\triangleright'} L)$, by using (51):

$$\begin{aligned} (1, z, w) (1, f, l) (1, z, w)^{-1} &= \beta(1, z, w) \bullet (1, f, l) \langle (1, z, w), (1, f, l)^{-1} \rangle^{-1} \\ &= (\beta(1, z, w) \bullet (1, f, l)) (1, 1, \{z\delta(w), (f\delta(l))^{-1}\}^{-1}). \end{aligned}$$

We thus have the following particular case of the lifted action Δ , see (79), which will be crucial later:

$$\begin{aligned} (g, x, 1, z, w) \Delta ((1, 1, 1), (1, f, l), (1, m)) &= \\ ((1, 1, 1), (g, xz) \bullet (1, f, l) (1, 1, (g\partial(x)) \triangleright \{z\delta(w), (f\delta(l))^{-1}\}^{-1}), (1, (g\partial(x)) \triangleright (\{z\delta(w), (f\delta(l))^{-1}\} \partial(z) \triangleright m)). \end{aligned}$$

In particular, the following subgroup $\overline{\text{Gr}_1(\text{Comp}(\mathcal{G}))}$ of $\text{Gr}_1(\text{Comp}(\mathcal{G}))$:

$$\overline{\text{Gr}_1(\text{Comp}(\mathcal{G}))} \doteq (\{1\} \ltimes_* (\{1\} \ltimes_{\triangleright'} \{1\})) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L)) \subset \text{Gr}_1(\text{Comp}(\mathcal{G}))$$

is closed under the action Δ of $\text{Gr}_0(\text{Comp}(\mathcal{G}))$ on $\text{Gr}_1(\text{Comp}(\mathcal{G}))$. We can also see that:

$$\begin{aligned} (g, x, 1, 1, 1) \Delta ((1, 1, 1), (1, f, l), (1, m)) &= ((1, 1, 1), (g, x) \bullet (1, f, l), (1, (g\partial(x)) \triangleright m)) \\ &= ((1, 1, 1), (g, x) \bullet (1, f, l), (g, x) \bullet (1, m)), \end{aligned} \tag{86}$$

as it should, by definition of the lifted action of a 2-crossed module.

We therefore have a morphism of $(G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{G}))$ -modules (for the actions Δ and $|$, respectively), namely:

$$((1, 1, 1), (1, f, l), (1, m)) \in \overline{\text{Gr}_1(\text{Comp}(\mathcal{G}))} \subset \text{Gr}_1(\text{Comp}(\mathcal{G})) \mapsto (1, f, lm) \in \{1\} \ltimes_* (E \ltimes_{\triangleright'} L) \subset \text{Gr}_1(\mathcal{P}_*(\mathcal{G})),$$

where $(g, x, 1, z, w) | (1, f, l) = (g, xz) \bullet (1, f, l) = r'(g, x, 1, z, w) \bullet (1, f, l)$, which actually is induced by the pre-crossed module map of (82). Also there is a $\text{Gr}_0(\text{Comp}(\mathcal{G}))$ -module morphism for the actions Δ and $@$, where:

$$((1, 1, 1), (1, f, l), (1, m)) \in \overline{\text{Gr}_1(\text{Comp}(\mathcal{G}))} \subset \text{Gr}_1(\text{Comp}(\mathcal{G})) \mapsto (1, f\delta(l), m) \in (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) \subset \text{Gr}_1(\mathcal{P}_*(\mathcal{G})),$$

and the action $@$ has the form:

$$(g, x, 1, z, w) @ (1, f, k) = (g\partial(x), z\delta(w)) \bullet (1, f, k) = v(g, z, 1, z, w) \bullet (1, f, k).$$

(This can be shown directly or by using the 2-crossed module morphism of (83).)

The 2-crossed module maps $\text{Pr}_1^{\mathcal{P}_*(\mathcal{G})}, \mathcal{P}_*(\text{Pr}_1^{\mathcal{G}}): \mathcal{P}_*(\mathcal{P}_*(\mathcal{G})) \rightarrow \mathcal{P}_*(\mathcal{G})$ of 2.4.1 give us two group morphisms (see 2.3.3):

$$\text{Gr}_0(\text{Comp}(\mathcal{G})) \ltimes_{\Delta} \text{Gr}_1(\text{Comp}(\mathcal{G})) \rightarrow \text{Gr}_0(\mathcal{P}_*(\mathcal{G})) \ltimes_{\bullet} \text{Gr}_1(\mathcal{P}_*(\mathcal{G})).$$

These restrict to groups maps:

$$\text{Gr}_0(\text{Comp}(\mathcal{G})) \ltimes_{\Delta} \overline{\text{Gr}_1(\text{Comp}(\mathcal{G}))} \rightarrow \text{Gr}_0(\mathcal{P}_*(\mathcal{G})) \ltimes_{\bullet} \text{Gr}_1(\mathcal{P}_*(\mathcal{G})),$$

factoring through $\text{Gr}_0(\text{Comp}(\mathcal{G})) \subset \text{Gr}_0(\mathcal{P}_*(\mathcal{G})) \ltimes_{\bullet} \text{Gr}_1(\mathcal{P}_*(\mathcal{G}))$, as maps:

$$\begin{aligned} ((G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))) \ltimes_{\Delta} ((\{1\} \ltimes_* (\{1\} \ltimes_{\triangleright'} \{1\})) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L))) \\ \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{G})) \subset \text{Gr}_0(\mathcal{P}_*(\mathcal{G})) \ltimes_{\bullet} \text{Gr}_1(\mathcal{P}_*(\mathcal{G})). \end{aligned}$$

They have the form (respectively):

$$((g, x, 1, z, w), (1, 1, 1, 1, f, l, 1, m)) \mapsto (g, xz, 1, f, lm), \tag{87}$$

$$((g, x, 1, z, w), (1, 1, 1, 1, f, l, 1, m)) \mapsto (g\partial(x), z\delta(w), 1, f\delta(l), m). \tag{88}$$

We can prove directly that these are group morphisms, by the calculations above. We also clearly have an additional group morphism $\text{Gr}_0(\text{Comp}(\mathcal{G})) \ltimes_{\Delta} \overline{\text{Gr}_1(\text{Comp}(\mathcal{G}))} \rightarrow \text{Gr}_0(\text{Comp}(\mathcal{G}))$ namely:

$$((g, x, 1, z, w), (1, 1, 1, 1, f, l, 1, m)) \mapsto (g, x, 1, z, w), \tag{89}$$

These group morphisms will have a prime importance later, for instance in proving that the concatenation of homotopies is associative.

We note that $\text{Gr}_0(\text{Comp}(\mathcal{G})) \ltimes_{\Delta} \overline{\text{Gr}_1(\text{Comp}(\mathcal{G}))}$ can be seen as being the underlying group of the tetrahedron space 2-crossed module of \mathcal{G} . The previous three morphisms of course denote restrictions to three of the faces of the tetrahedron. The restriction to the fourth face is the morphism in (108).

2.5.3 Two embeddings of the composer pre-crossed module

Despite its apparently complicated definition, the composer pre-crossed module is very simple to deal with given that it is included in the direct product of much simpler pre-crossed modules. For the definition of $\eta = \eta_{\text{Comp}(\mathcal{G})}$ see example 2. Recall that we have a pre-crossed module $\eta(\text{Comp}(\mathcal{G})) = (\text{Gr}_0(\text{Comp}(\mathcal{G})) \rightarrow \text{Gr}_0(\text{Comp}(\mathcal{G})))$, given by the identity map $\text{Gr}_0(\text{Comp}(\mathcal{G})) \rightarrow \text{Gr}_0(\text{Comp}(\mathcal{G}))$ and the adjoint action, as well as a pre-crossed module map $\eta = (\beta', \text{id}): \text{Comp}(\mathcal{G}) \rightarrow \eta(\text{Comp}(\mathcal{G}))$. Recall that β' denotes the boundary map in $\text{Comp}(\mathcal{G})$, equation (80).

Lemma 21 *Let \mathcal{G} be a 2-crossed module. The pre-crossed module map $(\eta, \text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}, \mathcal{P}_*(\text{Pr}_1^{\mathcal{G}})): \text{Comp}(\mathcal{G}) \rightarrow \eta(\text{Comp}(\mathcal{G})) \times \mathcal{P}_*(\mathcal{G}) \times \mathcal{P}_*(\mathcal{G})$ is injective, and therefore, given another pre-crossed module $\mathcal{G}' = (\partial: E' \rightarrow G', \triangleright)$, a pair (Y, X) of (set) maps $(Y, X): \mathcal{G}' \rightarrow \text{Comp}(\mathcal{G})$, such that $\beta' \circ Y = X \circ \partial$, is a pre-crossed module map if, and only if, so is $(\eta, \text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}, \mathcal{P}_*(\text{Pr}_1^{\mathcal{G}})) \circ (Y, X)$.*

Proof. Easy calculations. ■

Analogously:

Lemma 22 *The pre-crossed module map $(\eta, \text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}, \text{Pr}_1^{\mathcal{P}_*(\mathcal{G})}): \text{Comp}(\mathcal{G}) \rightarrow \eta(\text{Comp}(\mathcal{G})) \times \mathcal{P}_*(\mathcal{G}) \times \mathcal{P}_*(\mathcal{G})$ is injective, and therefore, given another pre-crossed module $\mathcal{G}' = (\partial: E' \rightarrow G', \triangleright)$ a pair (Y, X) of (set) maps $(Y, X): \mathcal{G}' \rightarrow \text{Comp}(\mathcal{G})$, such that $\beta' \circ Y = X \circ \partial$, is a pre-crossed module map if, and only if, so is $(\eta, \text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}, \text{Pr}_1^{\mathcal{P}_*(\mathcal{G})}) \circ (Y, X)$.*

In particular, given that $\eta = (\beta', \text{id})$ it follows that (one more case is included):

Lemma 23 *Let \mathcal{G} be a 2-crossed module. Given a pre-crossed module \mathcal{G}' , a set-wise chain-map $f: \mathcal{G}' \rightarrow \text{Comp}(\mathcal{G})$ is a pre-crossed module map if, and only if, its underlying map on groups $\text{Gr}_0(\mathcal{G}') \rightarrow \text{Gr}_0(\text{Comp}(\mathcal{G}))$ is a group morphism and, moreover, the composition of f with two elements of the set $\{\text{Pr}_0^{\mathcal{P}_*(\mathcal{G})}, \text{Pr}_1^{\mathcal{P}_*(\mathcal{G})}, \mathcal{P}_*(\text{Pr}_1^{\mathcal{G}})\}$ defines a pre-crossed module map $\mathcal{G}' \rightarrow \mathcal{P}_*(\mathcal{G})$.*

3 Pointed homotopy of 2-crossed module maps

3.1 Quadratic derivations and 1-fold homotopy of 2-crossed modules

Suppose we have two pre-crossed modules $(E \rightarrow G, \triangleright)$ and $(E' \rightarrow G', \triangleright)$.

Definition 24 *Let $\phi: G \rightarrow G'$ be a group morphism. A ϕ -derivation $s: G \rightarrow E'$ is a set map such that, for each $g, h \in G$, we have:*

$$s(gh) = \phi(h)^{-1} \triangleright s(g) s(h).$$

Note that if $s: G \rightarrow E'$ is a derivation then:

$$s(1_G) = 1_{E'} \text{ and } s(g^{-1}) = \phi(g) \triangleright s(g)^{-1}. \quad (90)$$

Remark 25 *By looking at remark 12, ϕ -derivations are¹ in one-to-one correspondence with group maps $G \rightarrow G' \ltimes_{\triangleright} E'$, where $g \mapsto (\phi(g), s(g))$. In particular if G is free, a ϕ derivation $s: G \rightarrow E'$ can be specified (and uniquely) by its value on a free basis of G . We will very frequently use this line of thinking.*

Let

$$\mathcal{A} = \left(L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \} \right) \text{ and } \mathcal{A}' = \left(L' \xrightarrow{\delta} E' \xrightarrow{\partial} G', \triangleright, \{, \} \right)$$

be 2-crossed modules. Let also $f = (\mu, \psi, \phi): \mathcal{A} \rightarrow \mathcal{A}'$ be a 2-crossed module morphism.

Definition 26 *A pair (s, t) of maps $s: G \rightarrow E'$ and $t: E \rightarrow L'$ will be called a quadratic f -derivation: if s and t satisfy, for each $g, h \in G$ and $a, b \in E$:*

$$s(gh) = (\phi(h)^{-1} \triangleright s(g)) s(h), \quad (91)$$

$$t(ab) = (\psi(b)((s \circ \partial)(b)))^{-1} \triangleright' (\{\psi(b), \phi(\partial(b)^{-1}) \triangleright s(\partial(a))^{-1}\} t(a)) t(b), \quad (92)$$

¹This was pointed out to us by Ronnie Brown.

or alternatively, (12):

$$t(ab) = ((s \circ \partial)(b))^{-1} \triangleright' \left(\{\psi(b)^{-1}, s(\partial(a))^{-1}\}^{-1} \psi(b)^{-1} \triangleright' t(a) \right) t(b), \quad (93)$$

and also (for each $g \in G$ and $a \in E$):

$$\begin{aligned} t(g \triangleright a) &= \phi(g) \triangleright \left(s(g) s(\partial(a))^{-1} \triangleright' \left\{ \psi(a)^{-1}, s(g)^{-1} \right\}^{-1} \right) \\ &\quad \phi(g) \triangleright \left\{ s(g), s(\partial(a))^{-1} \psi(a)^{-1} \right\} (\phi(g) (\partial \circ s)(g)) \triangleright t(a). \end{aligned} \quad (94)$$

Lemma 27 (Pointed homotopy of 2-crossed module maps) *In the condition of the previous definition, if (s, t) is a quadratic f -derivation, and if we define: $f' = (\mu', \psi', \phi') : \mathcal{A} \rightarrow \mathcal{A}'$ as:*

$$\mu'(l) = \mu(l) (t \circ \delta)(l), \text{ where } l \in L \quad (95)$$

$$\psi'(a) = \psi(a) ((s \circ \partial))(a) (\delta \circ t)(a), \text{ where } a \in E \quad (96)$$

$$\phi'(g) = \phi(g) (\partial \circ s)(g), \text{ where } g \in G \quad (97)$$

then f' a morphism of 2-crossed modules $\mathcal{A} \rightarrow \mathcal{A}'$. In this case we put:

$$f \xrightarrow{(f, s, t)} f'$$

and say that (f, s, t) is a (1-fold) homotopy connecting f and f' .

This can be proved by using the following lemma, which has an immediate proof, and noting that $f' = \text{Pr}_1^{\mathcal{A}'} \circ H$ and also $f = \text{Pr}_0^{\mathcal{A}'} \circ H$; see theorem 17.

Lemma 28 *Given a 2-crossed module morphism $f = (\mu, \psi, \phi) : \mathcal{A} \rightarrow \mathcal{A}'$, the pair of maps $t : E \rightarrow L'$ and $s : G \rightarrow E'$ is a quadratic f -derivation if, and only if, $H = (i_3, i_2, i_1) : \mathcal{A} \rightarrow \mathcal{P}_*(\mathcal{A}')$ is a 2-crossed module morphism, where*

$$\begin{aligned} l \in L &\xrightarrow{i_3} (\mu(l), t \circ \delta(l)) \in L' \ltimes_{\text{ad}} L' \\ a \in E &\xrightarrow{i_2} (\psi(a), (s \circ \partial)(a), t(a)) \in E' \ltimes_* (E' \ltimes_{\triangleright'} L') \\ g \in G &\xrightarrow{i_1} (\phi(g), s(g)) \in G' \ltimes_{\triangleright} E'. \end{aligned}$$

Also, $H = (i_3, i_2, i_1)$ is a 2-crossed module map if, and only if, (i_2, i_1) is a pre-crossed module map.

Proof. Conditions (91), (92), (94) express exactly that (i_2, i_1) is a pre-crossed module morphism. That H defines a morphism of complexes $\mathcal{A} \rightarrow \mathcal{P}_*(\mathcal{A}')$, which is equivariant with respect to the actions of G and $G' \ltimes_{\triangleright} E'$, if and only if, (t, s) is a quadratic f -derivation follows from the explicit construction of $\mathcal{P}_*(\mathcal{A}')$. We now need to prove that H always preserves the Peiffer lifting, which follows immediately from the equation:

$$t(\langle a, b \rangle) = \{\psi(a), \psi(b)\}^{-1} \{ \psi(a) s(\partial(a)) \delta(t(a)), \psi(b) s(\partial(b)) \delta(t(b)) \}, \text{ for each } a, b \in E. \quad (98)$$

This equation was proven in [21]. ■

Remark 29 *Note that if (s, t) is a quadratic f derivation, where $f : \mathcal{A} \rightarrow \mathcal{A}'$, connecting f and f' , and if $g : \mathcal{A}' \rightarrow \mathcal{A}''$ is 2-crossed module map, then $(g \circ s, g \circ t)$ is a quadratic $(g \circ f)$ -derivation connecting $g \circ f$ and $g \circ f'$. Analogously if $h : \mathcal{A}''' \rightarrow \mathcal{A}$ is a 2-crossed module map then $(s \circ h, t \circ h)$ is a quadratic $(f \circ h)$ -quadratic derivation connecting $f \circ h$ and $f' \circ h$.*

3.2 Quadratic 2-derivations and 2-fold homotopy of 2-crossed modules

Let

$$\mathcal{A} = \left(L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \} \right) \text{ and } \mathcal{A}' = \left(L' \xrightarrow{\delta} E' \xrightarrow{\partial} G', \triangleright, \{, \} \right)$$

be 2-crossed modules. Let $f = (\mu, \psi, \phi) : \mathcal{A} \rightarrow \mathcal{A}'$ be a 2-crossed module morphism. Consider a quadratic f -derivation (s, t) . Recall the construction of the disk space $\mathcal{D}_*(\mathcal{A}')$ of \mathcal{A}' ; 2.4.2.

Definition 30 We say that a map $k: G \rightarrow L'$ is a quadratic (f, s, t) 2-derivation if for each $g, h \in G$:

$$k(gh) = \left(s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k(g)) \right) k(h). \quad (99)$$

Lemma 31 The map $k: G \rightarrow L'$ is a quadratic (f, s, t) 2-derivation if and only if $g \mapsto (\phi(g), s(g), k(g))$ is a group morphism $G \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} L = \text{Gr}_0(\mathcal{D}_*(\mathcal{A}'))$. In particular $k(1_G) = 1_K$ and for each $g \in G$:

$$k(g^{-1}) = \phi(g) \triangleright (s(g) \triangleright' k(g)^{-1}).$$

Moreover if $g, h, i \in G$:

$$k(g^{-1}hi) = \Xi^{(\phi, s, k)}(g, h, i),$$

where we have defined:

$$\Xi^{(\phi, s, k)}(g, h, i) = s(i)^{-1} \triangleright' \left(\phi(i)^{-1} \triangleright \left(s(h)^{-1} \triangleright' ((\phi(h)^{-1} \phi(g)) \triangleright (s(g) \triangleright' k(g)^{-1})) k(h) \right) \right) k(i). \quad (100)$$

Proof. The first assertion is immediate from remark 12 and the construction of the disk space, more precisely (66). Also:

$$\begin{aligned} k(g^{-1}hi) &= s(i)^{-1} \triangleright' \left(\phi(i)^{-1} \triangleright (s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k(g^{-1})) k(h)) \right) k(i) \\ &= s(i)^{-1} \triangleright' \left(\phi(i)^{-1} \triangleright (s(h)^{-1} \triangleright' (\phi(h^{-1}g) \triangleright (s(g) \triangleright' k(g)^{-1})) k(h)) \right) k(i) \end{aligned}$$

■

Lemma 32 The set map $k: G \rightarrow L'$ is a quadratic (f, s, t) 2-derivation if, and only if, the map $H^2 = (j_3, j_2, j_1)$, where:

$$\begin{aligned} j_1(g) &= (\phi(g), s(g), k(g)) \\ j_2(e) &= (\psi(e), ((s \circ \partial)(e)), t(e), (k \circ \partial)(e)) \\ j_3(l) &= (l, (t \circ \partial(l))) \end{aligned}$$

is a 2-crossed module morphism from \mathcal{A} into $\mathcal{D}_*(\mathcal{A}')$.

Proof. We already know by lemma 28 that, forgetting the last component of j_1 and j_2 , we have a 2-crossed module map $\mathcal{A} \rightarrow \mathcal{P}_*(\mathcal{A}')$. Given the form of $\mathcal{D}_*(\mathcal{A}')$ we clearly get a morphism of group complexes $\mathcal{A} \rightarrow \mathcal{D}_*(\mathcal{A}')$, which is compatible with the action of the first groups of the complexes on the remaining. To prove this note the following calculation, and compare with (69), (70), (66) and (71):

$$\begin{aligned} k(\partial(g \triangleright e)) &= k(g \partial(e) g^{-1}) = (\phi(g) \triangleright s(g)) \triangleright' (\phi(g) \triangleright k(g \partial(e))) \phi(g) \triangleright (s(g) \triangleright' k(g)^{-1}) \\ &= \phi(g) \triangleright \left(s(g) \triangleright' (k(g \partial(e)) k(g)^{-1}) \right) \\ &= \phi(g) \triangleright (s(g) \triangleright' \left((s(\partial(e))^{-1} \triangleright' (\phi(\partial(e))^{-1} \triangleright k(g)) \right) k(\partial(e)) k(g)^{-1} \right)). \end{aligned}$$

We now need to prove that H^2 preserves the Peiffer lifting. This follows from lemma 28 and the form of the Peiffer lifting on $\mathcal{D}_*(\mathcal{A}')$ and $\mathcal{P}_*(\mathcal{A}')$, see (52) and (72). ■

Therefore, looking at the maps $(p, q, r), (p', q', r'): \mathcal{D}_*(\mathcal{A}') \rightarrow \mathcal{P}_*(\mathcal{A}')$ of (73) and (74), lemma 28 and the previous one, given a quadratic (f, s, t) 2-derivation k then $s': G \rightarrow E'$ and $t': E \rightarrow L'$, defined as:

$$\begin{aligned} s'(g) &= s(g) (\delta \circ k)(g) \\ t'(e) &= (k \circ \partial)(e)^{-1} t(e) \end{aligned} \quad (101)$$

yield a quadratic f -derivation (s', t') , and in this case we put:

$$(f, s, t) \xrightarrow{(f, s, t, k)} (f, s', t').$$

It is not difficult to prove directly that (s', t') is an f -derivation. For example (93) follows from (in the third equality we use $\delta(z) \triangleright' w = zwz^{-1}$ and in the last equation (17)):

$$\begin{aligned}
t'(ab) &= k(\partial(a)\partial(b))^{-1} t(ab) \\
&= k(\partial(b))^{-1} \left(s(\partial(b))^{-1} \triangleright' (\phi(\partial(b))^{-1} \triangleright k(\partial(a))) \right)^{-1} \left((s \circ \partial)(b)^{-1} \triangleright' \{\psi(b)^{-1}, s(\partial(a))^{-1}\}^{-1} \right) \\
&\quad \left((\psi(b) (s \circ \partial)(b))^{-1} \triangleright' t(a) \right) t(b) \\
&= k(\partial(b))^{-1} s(\partial(b))^{-1} \triangleright' \left((\phi(\partial(b))^{-1} \triangleright k(\partial(a)))^{-1} \{\psi(b)^{-1}, s(\partial(a))^{-1}\}^{-1} \right) \left((\psi(b) (s \circ \partial)(b))^{-1} \triangleright' t(a) \right) t(b) \\
&= s'(\partial(b))^{-1} \triangleright' \left((\phi(\partial(b))^{-1} \triangleright k(\partial(a)))^{-1} \{\psi(b)^{-1}, s(\partial(a))^{-1}\}^{-1} \right) \left((\psi(b) (s' \circ \partial)(b))^{-1} \triangleright' t(a) \right) t'(b) \\
&= s'(\partial(b))^{-1} \triangleright' \left((\phi(\partial(b))^{-1} \triangleright k(\partial(a)))^{-1} \{\psi(b)^{-1}, s(\partial(a))^{-1}\}^{-1} \psi(b)^{-1} \triangleright' t(a) \right) t'(b) \\
&= s'(\partial(b))^{-1} \triangleright' \left(\{\psi(b)^{-1}, s'(\partial(a))^{-1}\}^{-1} \psi(b)^{-1} \triangleright' t'(a) \right) t'(b).
\end{aligned}$$

Remark 33 Note that if $f \xrightarrow{(f,s,t)} f'$ then we also have $f \xrightarrow{(f,s',t')} f'$.

Definition 34 We say that the quadratic f -derivations (s, t) and (s', t') are 2-fold homotopic if there exists a quadratic (f, s, t) 2-derivation such that $(f, s, t) \xrightarrow{(f,s,t,k)} (f, s', t')$. The quadruple (f, s, t, k) will be called a 2-fold homotopy connecting (f, s, t) and (f, s', t') .

Remark 35 Looking at the definition of the composer pre-crossed module in subsection 2.5, an (f, s, t) 2-derivation is a map $k \rightarrow L'$ such that $g \mapsto (\phi(g), s(g), 1, \delta(k(g)), k(g)^{-1})$ is a group morphism $G \rightarrow (G' \ltimes_{\triangleright} E') \ltimes_{\bullet} (\{1\} \ltimes_{*} (E' \ltimes_{\triangleright'} L')) = \text{Gr}_0(\text{Comp}(\mathcal{A}'))$. This follows from (77). (We have put $f = (\mu, \psi, \phi)$.)

3.3 A groupoid of 2-crossed module maps and their homotopies (in the free up to order one case)

In this subsection, let us fix two 2-crossed modules:

$$\mathcal{A}' = \left(L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \} \right) \text{ and } \mathcal{A} = \left(L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \} \right).$$

Suppose also that F is a free group, with a chosen (free) basis $B \subset F$. Let us define a groupoid $[\mathcal{A}', \mathcal{A}]_1^B$, with objects the 2-crossed module maps $\mathcal{A}' \rightarrow \mathcal{A}$ and morphisms their homotopies. This will explicitly depend on the chosen basis B of F . We will freely use the notation of subsections 2.3, 2.5 and 3.1.

3.3.1 Concatenating homotopies in the free up to order one case

Consider homotopies of 2-crossed module maps $\mathcal{A}' \rightarrow \mathcal{A}$:

$$f = (\mu, \psi, \phi) \xrightarrow{(f,s,t)} f' = (\mu', \psi', \phi') \text{ and } (\mu', \psi', \phi') \xrightarrow{(f',s',t')} f'' = (\mu'', \psi'', \phi'')$$

(where $f, f', f'': \mathcal{A}' \rightarrow \mathcal{A}$). Let us define their concatenation:

$$f = (\mu, \psi, \phi) \xrightarrow{(f,s \otimes s', t \otimes t')} f'' = (\mu'', \psi'', \phi'').$$

The derivation $(s \otimes s'): F \rightarrow E$ is the unique ϕ -derivation (see definition 24 and remark 25) which on the chosen basis B of F has the form

$$b \in B \mapsto (s \otimes s')(b) = s(b)s'(b) \in E.$$

There is another piece of information that we will use, namely a set map $\omega^{(s,s')}: F \rightarrow L$, measuring the difference (for each $g \in F$) between $s(g)s'(g)$ and $(s \otimes s')(g)$; this difference is null in a crossed module.

Recall the construction of the composer pre-crossed module 2.5. Given that F is free, there exists a unique group map $X: F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A}))$, say:

$$g \xrightarrow{X} (\phi(g), s(g), 1, \zeta(g), \omega^{(s,s')}(g))$$

which on the chosen basis B of F takes the form:

$$b \xrightarrow{X} (\phi(b), s(b), 1, s'(b), 1).$$

(We will usually abbreviate $\omega = \omega^{(s,s')}$.) In particular, by composing with the group map

$$r': (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A})) \rightarrow G \ltimes_{\triangleright} E = \text{Gr}_0(\mathcal{P}_*(\mathcal{A}))$$

in (82): (thus $r'(g, x, 1, e, k) = (g, xe)$) gives a group morphism $F \rightarrow G \ltimes_{\triangleright} E$, with

$$g \mapsto (\phi(g), s(g) \zeta(g))$$

for each g in F . The value of this morphism on generators of F is $b \mapsto (\phi(b), s(b)s'(b))$. In particular (by remark 25) it follows that for each $g \in F$:

$$(s \otimes s')(g) = s(g) \zeta(g). \quad (102)$$

On the other hand, we have another group map $v: (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A})) \rightarrow G \ltimes_{\triangleright} E = \text{Gr}_0(\mathcal{P}_*(\mathcal{A}))$ (see (63) and (83)), where $v(g, x, 1, e, k) = (g\partial(x), e\delta(k))$. By composing the map $X: F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L))$ with v , gives a group map $F \rightarrow G \ltimes_{\triangleright} E$, of the form $g \mapsto (\phi'(g), \zeta(g)\delta(\omega(g)))$, which on generators is $b \mapsto (\phi'(b), s'(b))$, having therefore the form (by remark 25) $g \in F \mapsto (\phi'(g), s'(g))$. In particular it follows that $\zeta(g)\delta(\omega(g)) = s'(g)$, or

$$s(g)\zeta(g)\delta(\omega(g)) = s(g)s'(g), \text{ for each } g \in F. \quad (103)$$

We have proven:

Lemma 36 *There exists a unique group morphism $X: F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A}))$ with*

$$g \mapsto (\phi(g), s(g), 1, s'(g)\delta(\omega^{(s,s')}(g))^{-1}, \omega^{(s,s')}(g)),$$

for each g in F , such that $\omega(b) = 1$ on the free generators $b \in B$ of F . In particular by (102) and (103)

$$(s \otimes s')(g) = s(g) s'(g) \delta(\omega^{(s,s')}(g))^{-1}, \text{ for each } g \in F. \quad (104)$$

Remark 37 *Clearly*

$$\omega^{(s,s')}(1_F) = 1_L.$$

For each $b \in B$ we have

$$\omega^{(s,s')}(b) = 1_L.$$

Also, by remark 12 and equation (76):

$$\omega^{(s,s')}(gh) = \omega^{(s,s')}(h) s'(h)^{-1} \triangleright' \left(\phi(h)^{-1} \triangleright \{ \phi(h) \triangleright s(h)^{-1}, \delta(\omega^{(s,s')}(g)) s'(g)^{-1} \} \phi'(h)^{-1} \triangleright \omega^{(s,s')}(g) \right)$$

and

$$\omega^{(s,s')}(g^{-1}) = \phi(g) \triangleright \{ s(g), s'(g)(\omega^{(s,s')}(g))^{-1} \} \phi'(g) \triangleright (s'(g) \triangleright' (\omega^{(s,s')}(g))^{-1}).$$

In particular, if b, b', b'' are free generators of F :

$$\omega^{(s,s')}(bb') = s'(b')^{-1} \triangleright' (\{ s(b')^{-1}, \phi(b')^{-1} \triangleright s'(b)^{-1} \}),$$

$$\omega^{(s,s')}(b^{-1}) = \phi(b) \triangleright \{ s(b), s'(b) \},$$

and

$$\omega^{(s,s')}(b^{-1}b'b'')$$

$$\begin{aligned}
&= \omega^{(s,s')}(b'b'')^{-1} \triangleright' \left(\phi(b'b'')^{-1} \triangleright \{ \phi(b'b'') \triangleright s(b'b'')^{-1}, \delta(\omega^{(s,s')}(b^{-1}))s'(b^{-1})^{-1} \} \phi'(b'b'')^{-1} \triangleright \omega^{(s,s')}(b^{-1}) \right) \\
&= s'(b'')^{-1} \triangleright' (\{s(b'')^{-1}, \phi(b'')^{-1} \triangleright s'(b')^{-1}\}) \\
&\quad s'(b'b'')^{-1} \triangleright' \left(\phi(b'b'')^{-1} \triangleright \{ \phi(b'b'') \triangleright s(b'b'')^{-1}, \delta(\omega^{(s,s')}(b^{-1}))s'(b^{-1})^{-1} \} \phi'(b'b'')^{-1} \triangleright \omega^{(s,s')}(b^{-1}) \right) \\
&= s'(b'')^{-1} \triangleright' (\{s(b'')^{-1}, \phi(b'')^{-1} \triangleright s'(b')^{-1}\}) \\
&\quad s'(b'b'')^{-1} \triangleright' \left(\phi(b'b'')^{-1} \triangleright \{ \phi(b'b'') \triangleright s(b'b'')^{-1}, \delta(\omega^{(s,s')}(b^{-1}))s'(b^{-1})^{-1} \} (\phi'(b'b'')^{-1} \phi(b)) \triangleright \{s(b), s'(b)\} \right) \\
&= s'(b'')^{-1} \triangleright' (\{s(b'')^{-1}, \phi(b'')^{-1} \triangleright s'(b')^{-1}\}) \\
&\quad s'(b'b'')^{-1} \triangleright' \left(\phi(b'b'')^{-1} \triangleright \{ \phi(b'b'') \triangleright s(b'b'')^{-1}, \phi(b) \triangleright (s(b) s'(b) s(b)^{-1}) \} (\phi'(b'b'')^{-1} \phi(b)) \triangleright \{s(b), s'(b)\} \right) \\
&= s'(b'')^{-1} \triangleright' (\{s(b'')^{-1}, \phi(b'')^{-1} \triangleright s'(b')^{-1}\}) \\
&\quad s'(b'b'')^{-1} \triangleright' \left(\{s(b'b'')^{-1}, (\phi(b'b'')^{-1} \phi(b)) \triangleright (s(b) s'(b) s(b)^{-1}) \} (\phi'(b'b'')^{-1} \phi(b)) \triangleright \{s(b), s'(b)\} \right).
\end{aligned}$$

Thus:

$$\omega^{(s,s')}(b^{-1}b'b'') = \Theta^{(s,s')}(b, b', b''),$$

where by definition:

$$\begin{aligned}
\Theta^{(s,s')}(b, b', b'') &= s'(b'')^{-1} \triangleright' (\{s(b'')^{-1}, \phi(b'')^{-1} \triangleright s'(b')^{-1}\}) \\
&\quad (\phi'(b'')^{-1} \triangleright s'(b') s'(b''))^{-1} \triangleright' \left(\left\{ (\phi(b'')^{-1} \triangleright s(b') s(b''))^{-1}, (\phi(b'b'')^{-1} \phi(b)) \triangleright (s(b) s'(b) s(b)^{-1}) \right\} \right. \\
&\quad \left. (\phi'(b'b'')^{-1} \phi(b)) \triangleright \{s(b), s'(b)\} \right). \quad (105)
\end{aligned}$$

We now define $(t \otimes t'): E' \rightarrow L$. Put $\omega = \omega^{(s,s')}$. For $e \in E'$ put:

$$(t \otimes t')(e) = ((\omega \circ \partial)(e)) (s'(\partial(e))^{-1} \triangleright' t(e)) \quad t'(e) = (\omega^{(s,s')}(e)) (s'(\partial(e))^{-1} \triangleright' t(e)) \quad t'(e).$$

The complicated bit is to prove that $(s \otimes s', t \otimes t')$ is indeed a quadratic (μ, ψ, ϕ) -derivation. Let us see how this can be proven: Looking again at the construction of the composer pre-crossed module (subsection 2.5), consider the set map:

$$Y: E' \rightarrow (E \rtimes_* (E \rtimes_{\triangleright'} L)) \rtimes_* ((\{1\} \rtimes_* (E \rtimes_{\triangleright'} L)) \rtimes_{\bullet'} (\{1\} \rtimes_{\text{ad}} L)) = \text{Gr}_1(\text{Comp}(\mathcal{A})),$$

definition 3, of the form:

$$e \xrightarrow{Y} (\psi(e), s(\partial(e)), t(e), 1, s'(\partial(e))\delta(\omega(\partial(e)))^{-1}, \omega(\partial(e)), 1, t'(e)).$$

Note that $\beta' \circ Y = X \circ \partial$, where $\beta: \text{Gr}_1(\text{Comp}(\mathcal{A})) \rightarrow \text{Gr}_0(\text{Comp}(\mathcal{A}))$ is the boundary map, (80). Let us prove that we have a pre-crossed module map $(Y, X): (\partial: E' \rightarrow F) \rightarrow \text{Comp}(\mathcal{A})$. This done by using lemma 21, in the form of lemma 23: The composition of (Y, X) with $\text{Pr}_0^{\mathcal{P}_*(\mathcal{A})}$ is:

$$g \in F \mapsto (\phi(g), s(g)) \in \text{Gr}_0(\mathcal{P}_*(\mathcal{A})) \text{ and } e \in E' \mapsto (\psi(e), s(\partial(e)), t(e)) \in \text{Gr}_1(\mathcal{P}_*(\mathcal{A})),$$

a pre-crossed module map by lemma 28; the composition of (Y, X) with $\mathcal{P}_*(\text{Pr}_1^{\mathcal{A}})$ is:

$$g \in F \mapsto (\phi'(g), s'(g)) \in \text{Gr}_0(\mathcal{P}_*(\mathcal{A})) \text{ and } e \in E' \mapsto (\psi'(e), s'(\partial(e)), t'(e)) \in \text{Gr}_1(\mathcal{P}_*(\mathcal{A})),$$

again a pre-crossed module map by lemma 28; the underlying map $F \rightarrow \text{Gr}_0(\text{Comp}(\mathcal{A}))$ is

$$g \xrightarrow{X} (\phi(g), s(g), 1, s'(g)\delta(\omega(g))^{-1}, \omega(g))$$

and X is a group morphism, by lemma 36.

By composing (Y, X) with the pre-crossed module map $\text{Pr}_1^{\mathcal{P}_*(\mathcal{A})}$ of (82), yields a pre-crossed module map (Y', X') from $(E' \rightarrow F)$ to the underlying pre-crossed module of $\mathcal{P}_*(\mathcal{A})$. This has the form:

$$\begin{aligned}
g &\xrightarrow{X'} (\phi(g), s(g) s'(g) \delta(\omega(g))^{-1}) = (\phi(g), (s \otimes s')(g)) \\
e &\xrightarrow{Y'} \left(\psi(e), s(\partial(e)) s'(\partial(e)) \delta(\omega(\partial(e)))^{-1}, \omega(\partial(e)) (s'(\partial(e))^{-1} \triangleright' t(e)) \quad t'(e) \right) \\
&= \left(\psi(e), (s \otimes s')(\partial(e)), \omega(\partial(e)) (s'(\partial(e))^{-1} \triangleright' t(e)) \quad t'(e) \right).
\end{aligned}$$

(We have used the fact that $(\delta: L \rightarrow E, \triangleright')$ is a crossed module, thus $\delta(k) \triangleright' l = k l k^{-1}$.) By lemma 28 it follows the fundamental result that $((s \otimes s'), (t \otimes t'))$ is a (μ, ψ, ϕ) -quadratic derivation. Moreover:

Lemma 38 *Consider homotopies of 2-crossed module maps*

$$f = (\mu, \psi, \phi) \xrightarrow{(f, s, t)} f' = (\mu', \psi', \phi') \text{ and } (\mu', \psi', \phi') \xrightarrow{(f', s', t')} f'' = (\mu'', \psi'', \phi'').$$

Then the (μ, ψ, ϕ) -quadratic derivation $((s \otimes s'), (t \otimes t'))$ connects f and f'' .

Proof. Let $\bar{f} = (\bar{\mu}, \bar{\psi}, \bar{\phi})$ be the 2-crossed module morphism defined from f and the quadratic derivation $(s \otimes s', t \otimes t')$; lemma 27. We must prove that $\bar{f} = f''$. Let b be a free generator of F . Then

$$\bar{\phi}(b) = \phi(b) \partial(s \otimes s'(b)) = \phi(b) \partial(s(b) s'(b)) = \phi(b) \partial(s(b)) \partial(s'(b)) = \phi'(b) \partial(s'(b)) = \phi''(b).$$

In particular it follows that $\bar{\phi}(g) = \phi''(g)$, for each $g \in F$.

Given $e \in E'$ we have (we use (104)) and the rule $\delta(e \triangleright' k) = e \delta(k) e^{-1}$, for each $e \in E$ and $k \in L$:

$$\begin{aligned} \bar{\psi}(e) &= \psi(e) (s \otimes s')(\partial(e)) \delta((t \otimes t')(e)) \\ &= \psi(e) s(\partial(e)) s'(\partial(e)) \delta(\omega(\partial(e)))^{-1} \delta\left((\omega \circ \partial)(e) (s'(\partial(e))^{-1} \triangleright' t(e)) t'(e)\right) \\ &= \psi(e) s(\partial(e)) \delta(t(e)) s'(\partial(e)) \delta(t'(e)) = \psi'(e) s'(\partial(e)) \delta(t'(e)) = \psi''(e). \end{aligned}$$

Finally, given $k \in L'$ we have (since $\partial \circ \delta(k) = 1$ for each $k \in L$):

$$\bar{\mu}(k) = \mu(k) (t \otimes t')(\delta(k)) = \mu(k) t(\delta(k)) t'(\delta(k)) = \mu''(k).$$

Note $s'(1_F) = 1_E$ and $\omega(1_F) = 1_L$. ■

3.3.2 The concatenation of homotopies is associative (in the free up to order 1 case)

We freely use the notation of subsection 2.5, and we resume the notation and context of 3.3.1. The notation of 2.5.2 will be particularly important.

Proposition 39 *The concatenation of homotopies is associative.*

Proof. Choose a chain of homotopies of 2-crossed module maps $\mathcal{A}' \rightarrow \mathcal{A}$:

$$f = (\mu, \psi, \phi) \xrightarrow{(f, s, t)} f' = (\mu', \psi', \phi') \xrightarrow{(f', s', t')} f'' = (\mu'', \psi'', \phi'') \xrightarrow{(f'', s'', t'')} f''' = (\mu''', \psi''', \phi''').$$

It is immediate that $(s \otimes s') \otimes s'' = s \otimes (s' \otimes s'')$ since this is true in a free basis of F .

Let us now see that $(t \otimes t') \otimes t'' = t \otimes (t' \otimes t'')$. Put, for each $e \in E'$:

$$\begin{aligned} (t \otimes t')(e) &= ((\omega^{(s, s')} \circ \partial)(e)) (s'(\partial(e))^{-1} \triangleright' t(e)) t'(e), \\ (t' \otimes t'')(e) &= ((\omega^{(s', s'')} \circ \partial)(e)) (s''(\partial(e))^{-1} \triangleright' t'(e)) t''(e), \\ (t \otimes (t' \otimes t''))(e) &= (\omega^{(s, s' \otimes s'')} \circ \partial)(e) (s' \otimes s'')(\partial(e))^{-1} \triangleright' t(e) ((\omega^{(s', s'')} \circ \partial)(e)) (s''(\partial(e))^{-1} \triangleright' t'(e)) t''(e) \\ &= (\omega^{(s, s' \otimes s'')} \circ \partial)(e) \omega^{(s', s'')}(\partial(e)) (s'(\partial(e)) s''(\partial(e))^{-1} \triangleright' t(e) (s''(\partial(e))^{-1} \triangleright' t'(e)) t''(e), \\ ((t \otimes t') \otimes t'')(e) &= (\omega^{(s \otimes s', s'')} \circ \partial)(e) s''(\partial(e))^{-1} \triangleright' \left(((\omega^{(s, s')} \circ \partial)(e)) (s'(\partial(e))^{-1} \triangleright' t(e)) t'(e) \right) t''(e). \end{aligned}$$

To prove associativity, we therefore need to prove that for each $e \in E'$:

$$\omega^{(s, s' \otimes s'')}(\partial(e)) \omega^{(s', s'')}(\partial(e)) = \omega^{(s \otimes s', s'')}(\partial(e)) s''(\partial(e))^{-1} \triangleright' \omega^{(s, s')}(\partial(e)).$$

We will prove that for each $g \in F$ we have:

$$\omega^{(s, s' \otimes s'')} (g) = \omega^{(s \otimes s', s'')} (g) (s''(g)^{-1} \triangleright' \omega^{(s, s')} (g)) \omega^{(s', s'')} (g)^{-1}. \quad (106)$$

(It is a nice exercise to prove that this is coherent with (104).) To prove (106), consider the unique group map:

$$\begin{aligned} W: F &\rightarrow ((G \ltimes_\triangleright E) \ltimes_\bullet (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))) \ltimes_\Delta (\{1\} \ltimes_* (\{1\} \ltimes_{\triangleright'} \{1\}) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L))) \\ &= \text{Gr}_0(\text{Comp}(\mathcal{A})) \ltimes_\Delta \overline{\text{Gr}_1(\text{Comp}(\mathcal{A}))} \end{aligned} \quad (107)$$

(note that latter is a subgroup of $\text{Gr}_0(\text{Comp}(\text{Comp}(\mathcal{A})))$, which on the (chosen) free basis B of F is:

$$W(b) = (\phi(b), s(b), 1, s'(b), 1, 1, 1, 1, s''(b), 1, 1, 1).$$

By using the morphisms in (89), (87) and (88) and also lemma 36 of 3.3.1 we have that, for each $g \in F$ (by looking at the value of the compositions of these morphisms with W , in the chosen free basis of F):

$$\begin{aligned} W(g) &= \left(\phi(g), s(g), 1, s'(g) \delta(\omega^{(s,s')}(g))^{-1}, \omega^{(s,s')}(g), 1, 1, 1, 1, \right. \\ &\quad \left. s''(g) \delta(\omega^{(s \otimes s', s'')}(g))^{-1}, \omega^{(s \otimes s', s'')}(g) (\omega^{(s', s'')}(g))^{-1}, 1, \omega^{(s', s'')}(g) \right). \end{aligned}$$

In particular by composing with the group map below (derived from (60) in the context of the composer pre-crossed module):

$$\begin{aligned} (g, e, 1, f, l, 1, 1, 1, 1, x, k, 1, m) &\in \text{Gr}_0(\text{Comp}(\mathcal{A})) \ltimes_\Delta \overline{\text{Gr}_1(\text{Comp}(\mathcal{A}))} \subset \text{Gr}_0(\text{Comp}(\mathcal{A})) \ltimes_\Delta \text{Gr}_1(\text{Comp}(\mathcal{A})) \\ &\mapsto (g, e, 1, f, l) \beta'(1, 1, 1, 1, x, k, 1, m) \\ &= (g, e, 1, f, l)(1, 1, 1, x, k) \\ &= (g, e, 1, f, x, x^{-1} \triangleright' l, k) \in (G \ltimes_\triangleright E) \ltimes_\bullet (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A})), \end{aligned} \quad (108)$$

yields a group map

$$Z: F \rightarrow (G \ltimes_\triangleright E) \ltimes_\bullet (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A})), \quad (109)$$

which on the free generators $b \in B \subset F$ has the value:

$$Z(b) = (\phi(b), s(b), 1, s'(b)s''(b), 1),$$

and whose general value (for each $g \in F$) is:

$$Z(g) = (\phi(g), s(g), 1, s'(g) \delta(\omega^{(s,s')}(g))^{-1} s''(g) \delta(\omega^{(s \otimes s', s'')}(g))^{-1}, \omega^{(s \otimes s', s'')}(g) (s''(g))^{-1} \triangleright' \omega^{(s,s')}(g) \omega^{(s', s'')}(g)^{-1}).$$

On the other hand on generators $b \in B$ we also have:

$$Z(b) = (\phi(b), s(b), 1, (s' \otimes s'')(b), 1),$$

thus (by using lemma 36 again) for each $g \in F$:

$$Z(g) = (\phi(g), s(g), 1, (s' \otimes s'')(g) \delta(\omega^{(s, s' \otimes s'')}(g))^{-1}, \omega^{(s, s' \otimes s'')}(g)),$$

from which we have $\omega^{(s, s' \otimes s'')}(g) = \omega^{(s \otimes s', s'')}(g) s''(g)^{-1} \triangleright' \omega^{(s,s')}(g) \omega^{(s', s'')}(g)^{-1}$, for each $g \in F$. ■

3.3.3 Existence of units

Consider a homotopy between 2-crossed module maps:

$$f = (\mu, \psi, \phi) \xrightarrow{(f, s, t)} f' = (\mu', \psi', \phi').$$

Let (s_0^f, t_0^f) be the trivial quadratic f -derivation, i.e. such that $s_0^f(g) = 1$ for each $g \in F$ and $t_0^f(e) = 1$, for each $e \in E'$.

Lemma 40 *We have that*

$$\omega^{(s_0^f, s)}(g) = 1 \text{ and } \omega^{(s, s_0^{f'})}(g) = 1, \text{ for each } g \in F.$$

Proof. Consider the unique group map

$$X': F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A})),$$

of lemma 36, with, for each free generator $b \in B \subset F$:

$$X'(b) = (\phi(b), s_0^f(b), 1, s(b), 1).$$

Thus for each $g \in F$:

$$X'(g) = (\phi(g), s_0^f(g), 1, s(g)\delta(\omega^{(s_0^f, s)}(g))^{-1}, \omega^{(s_0^f, s)}(g)).$$

Since $g \in F \mapsto (\phi(g), 1, 1, s(g), 1) \in \text{Gr}_0(\text{Comp}(\mathcal{A}))$ is also a group morphism, by (35) and remark 25, which extends the value of X' in B , it follows in particular that $\omega^{(s_0^f, s)}(g) = 1$ for each $g \in F$.

Similarly, consider the map $X: F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))$ of lemma 36 with $s' = s_0^{f'}$. On free generators $X(b) = (\phi(b), s(b), 1, 1, 1)$. Since we have a group morphism with $g \in F \mapsto (\phi(g), s(g), 1, 1, 1) \in \text{Gr}_0(\text{Comp}(\mathcal{A}))$, there follows that $X(g) = (\phi(g), s(g), 1, 1, 1)$. In particular $\omega^{(s, s_0^{f'})}(g) = 1$, for each $g \in F$. ■

It therefore follows that $(s, t) \otimes (s_0^{f'}, t_0^{f'}) = (s, t)$ and that $(s_0^f, t_0^f) \otimes (s, t) = (s, t)$.

3.3.4 Inverting homotopies in the free up to order one case

We freely use the notation of subsection 2.5. As before, the notation of 2.5.2 will be particularly important. Let $f, f': \mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \}) \rightarrow \mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be 2-crossed module maps. Consider a 2-crossed module homotopy:

$$f = (\mu, \psi, \phi) \xrightarrow{(f, s, t)} f' = (\mu', \psi', \phi').$$

Let us define its inverse:

$$f' = (\mu', \psi', \phi') \xrightarrow{(f', \bar{s}, \bar{t})} f = (\mu, \psi, \phi).$$

The derivation $\bar{s}: F \rightarrow E$ is the unique ϕ' -derivation which on the chosen basis B of F takes the form:

$$\bar{s}(b) = s(b)^{-1}.$$

We clearly have $(s \otimes \bar{s})(g) = 1$ for each g in F , for this is true in a free basis of F , remark 25. Looking at equation (104), for each $g \in F$ we thus have

$$\bar{s}(g) = (s(g))^{-1} \delta(\omega^{(s, \bar{s})}). \quad (110)$$

Lemma 41 *For each $g \in F$ we have $\omega^{(\bar{s}, s)}(g) = s(g)^{-1} \triangleright' \omega^{(s, \bar{s})}(g)$.*

Proof. Consider the map:

$$\begin{aligned} W: G &\rightarrow ((G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))) \ltimes_{\Delta} (\{1\} \ltimes_* (\{1\} \ltimes_{\triangleright'} \{1\})) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L)) \\ &= \text{Gr}_0(\text{Comp}(\mathcal{A})) \ltimes_{\Delta} \overline{\text{Gr}_1(\text{Comp}(\mathcal{A}))} \end{aligned}$$

in equation (107), for $s' = \bar{s}$ and $s'' = s$. Then, since $(s \otimes \bar{s})(g) = 1$, for each $g \in F$, and lemma 40:

$$W(g) = \left(\phi(g), s(g), 1, s(g)^{-1}, \omega^{(s, \bar{s})}(g), 1, 1, 1, s(g), (\omega^{(\bar{s}, s)}(g))^{-1}, 1, \omega^{(\bar{s}, s)}(g) \right).$$

By composing with the group map of (108), gives a group morphism:

$$g \in F \xrightarrow{A} (\phi(g), s(g), 1, 1, (s(g)^{-1} \triangleright' \omega^{(s, \bar{s})}(g)) \omega^{(\bar{s}, s)}(g)^{-1}) \in (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)),$$

which on the chosen basis B of F has the form:

$$b \xrightarrow{A_0} (\phi(b), s(b), 1, 1, 1) \in (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)).$$

Since the map:

$$g \in F \mapsto (\phi(g), s(g), 1, 1, 1) \in (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))$$

is a group morphism extending A_0 it follows that $A(g) = (\phi(g), s(g), 1, 1, 1)$, thus:

$$(s(g)^{-1} \triangleright' \omega^{(s, \bar{s})}(g)) \omega^{(\bar{s}, s)}(g)^{-1} = 1,$$

for each $g \in F$.

■
We now define $\bar{t}: E' \rightarrow L$. For an $e \in E'$, put:

$$\bar{t}(e) = (\omega^{(s, \bar{s})}(\partial(e)))^{-1} (s\partial(e)) \triangleright' t(e)^{-1}. \quad (111)$$

Lemma 42 *The pair (\bar{s}, \bar{t}) is an f' -quadratic derivation.*

Proof. Consider the group map (lemma 36) $M: F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A}))$ such that

$$g \xrightarrow{M} (\phi(g), s(g), 1, \bar{s}(g)\delta(\omega^{(s, \bar{s})}(g))^{-1}, \omega^{(s, \bar{s})}(g)) = (\phi(g), s(g), 1, (s(g))^{-1}, \omega^{(s, \bar{s})}(g)).$$

Consider also the set map $N: E' \rightarrow (E \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L)) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L)) = \text{Gr}_1(\text{Comp}(\mathcal{A}))$ of the form:

$$e \xrightarrow{N} (\psi(e), s(\partial(e)), t(e), 1, (s(\partial(e)))^{-1}, \omega^{(s, \bar{s})}(\partial(e)), 1, (\omega^{(s, \bar{s})}(\partial(e)))^{-1} (s(\partial(e)) \triangleright' t(e)^{-1})).$$

Let us see that N is a group morphism and also that (N, M) is a pre-crossed module map $(\partial: E' \rightarrow F, \triangleright) \rightarrow \text{Comp}(\mathcal{A})$. We will use lemma 23. First note that $\beta' \circ N = M \circ \partial$. The composition of (N, M) with $\text{Pr}_0^{\mathcal{P}_*(\mathcal{A})}$ is, by equation (81):

$$g \in F \mapsto (\phi(g), s(g)) \in \text{Gr}_0(\mathcal{P}_*(\mathcal{A})) \text{ and } e \in E' \mapsto (\psi(e), s(\partial(e)), t(e)) \in \text{Gr}_1(\mathcal{P}_*(\mathcal{A}))$$

a pre-crossed module map by lemma 28. The composition of (N, M) with $\text{Pr}_1^{\mathcal{P}_*(\mathcal{A})}$ is, by equation (82):

$$g \in F \mapsto (\phi(g), 1) \in \text{Gr}_0(\mathcal{P}_*(\mathcal{A})) \text{ and } e \in E' \mapsto (\psi(e), 1, 1) \in \text{Gr}_1(\mathcal{P}_*(\mathcal{A}))$$

again pre-crossed module map by lemma 28, corresponding to the trivial quadratic derivation.

Therefore (N, M) is a pre-crossed module map. By composing (N, M) with the pre-crossed module map $\mathcal{P}_*(\text{Pr}_1^{\mathcal{A}})$ from $\text{Comp}(\mathcal{A})$ to the underlying crossed module of $\mathcal{P}_*(\mathcal{A})$ (see (83)), yields a pre-crossed module map from $(\partial: E' \rightarrow F, \triangleright)$ to the underlying pre-crossed module of $\mathcal{P}_*(\mathcal{A})$, which has the form:

$$\begin{aligned} e \in E' \mapsto & \left(\psi(e) s(\partial(e)) \delta(t(e)), (s(\partial(e)))^{-1} \delta(\omega^{(s, \bar{s})}(\partial(e))), \omega^{(s, \bar{s})}(\partial(e))^{-1} (s\partial(e)) \triangleright' t(e)^{-1} \right) \\ & = (\psi'(e), \bar{s}(\partial(e)), \bar{t}(e)) \in \text{Gr}_1(\mathcal{P}_*(\mathcal{A})) \end{aligned}$$

and

$$g \in F \mapsto (\phi(g) \partial(s(g)), \bar{s}(g)) = (\phi'(g), \bar{s}(g)) \in \text{Gr}_0(\mathcal{P}_*(\mathcal{A})),$$

hence (by lemma 28) it follows that (\bar{s}, \bar{t}) an f' -quadratic derivation. ■

Now note that obviously $s \otimes \bar{s} = s_0^f$ and $\bar{s} \otimes s = s_0^{f'}$, since the same is true in a free basis of F . On the other hand (by (110)), if $e \in E'$:

$$\begin{aligned} (t \otimes \bar{t})(e) &= \omega^{(s, \bar{s})}(\partial(e)) (\bar{s}(\partial(e))^{-1} \triangleright' t(e)) (\omega^{(s, \bar{s})}(\partial(e)))^{-1} s(\partial(e)) \triangleright' t(e)^{-1} \\ &= \omega^{(s, \bar{s})}(\partial(e)) ((\delta(\omega^{(s, \bar{s})}(\partial(e)))^{-1} s(\partial(e))) \triangleright' t(e)) (\omega^{(s, \bar{s})}(\partial(e)))^{-1} s(\partial(e)) \triangleright' t(e)^{-1} = 1, \end{aligned}$$

where we used the crossed module rule $\delta(k) \triangleright' l = k l k^{-1}$. Also (we use (110) again):

$$\begin{aligned} (\bar{t} \otimes t)(e) &= \omega^{(\bar{s}, s)}(\partial(e)) s^{-1}(\partial(e)) \triangleright' \bar{t}(\partial(e)) t(e) \\ &= s(\partial(e))^{-1} \triangleright' \omega^{(s, \bar{s})}(\partial(e)) (s(\partial(e)))^{-1} \triangleright' \left((\omega^{(s, \bar{s})}(\partial(e)))^{-1} (s(\partial(e)) \triangleright' t(e)^{-1}) \right) t(e) = 1. \end{aligned}$$

Thus we proved that (\bar{s}, \bar{t}) is an inverse of (s, t) .

We have therefore finished proving the main result of this subsection:

Theorem 43 Let $\mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \})$ and $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be 2-crossed modules, where F is a free group, with a chosen basis B . We can define a groupoid $[\mathcal{A}', \mathcal{A}]_1^B$ of 2-crossed module maps $\mathcal{A}' \rightarrow \mathcal{A}$, and their homotopies.

In the next subsection we will see that this construction can be expanded to be a 2-groupoid $\text{HOM}_B(\mathcal{A}', \mathcal{A})$, by considering 2-fold homotopies between 2-crossed module homotopies.

Corollary 44 Let \mathcal{A} and \mathcal{A}' be 2-crossed modules. If \mathcal{A}' is free up to order one then homotopy between 2-crossed module maps $\mathcal{A}' \rightarrow \mathcal{A}$ yields an equivalence relation.

3.4 A 2-groupoid of 2-crossed module maps, their homotopies and 2-fold homotopies (in the free up to order one case)

For the definition of a 2-groupoid see [29].

3.4.1 A groupoid of 2-crossed module homotopies and their 2-fold homotopies

Let $\mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} G', \triangleright, \{, \})$ and $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be 2-crossed modules. (In this subsection, only, we will not need to suppose that G' is a free group). Consider two 2-crossed module maps $f, f': \mathcal{A}' \rightarrow \mathcal{A}$. Let us define a groupoid $[f, f']$, with objects the homotopies (f, s, t) connecting f and f' (so the pair (s, t) is a quadratic f -derivation), the 2-morphisms being constructed from 2-fold homotopies (quadratic 2-derivations) $k: G' \rightarrow L$. For nomenclature and notation we refer to subsection 3.2.

The set of object of $[f, f']$ is the set of triples (f, s, t) where (s, t) is a quadratic f -derivation connecting f and f' . The set of 1-morphisms $(f, s, t) \rightarrow (f, s', t')$ is made out of quadruples (f, s, t, k) , where $k: G' \rightarrow L$ is a quadratic (f, s, t) 2-derivation, such that $(f, s, t) \xrightarrow{(f, s, t, k)} (f, s', t')$. If we have a chain of arrows:

$$(f, s, t) \xrightarrow{(f, s, t, k)} (f, s', t') \xrightarrow{(f, s', t', k')} (f, s'', t''),$$

then their concatenation is given by the map $k \diamond k': G' \rightarrow L$, such that:

$$(k \diamond k')(g) = k(g)k'(g)$$

for each $g \in G'$.

Lemma 45 The map $(k \diamond k'): G' \rightarrow L$ is a quadratic (f, s, t) 2-derivation.

Proof. By equations (99) and 101, and since $(\delta: L \rightarrow E, \triangleright')$ is a crossed module, we have, for each $g, h \in G'$:

$$\begin{aligned} (k \diamond k')(gh) &\doteq k(gh)k'(gh) \\ &= \left(s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k(g)) \right) k(h) \left(s'(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k'(g)) \right) k'(h) \\ &= \left(s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k(g)) \right) k(h) \left((\delta(k(h))^{-1} s(h)^{-1}) \triangleright' (\phi(h)^{-1} \triangleright k'(g)) \right) k'(h) \\ &= \left(s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k(g)) \right) \left(s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k'(g)) \right) k(h) k'(h). \\ &= \left(s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright (k \diamond k')(g)) \right) (k \diamond k')(h). \end{aligned}$$

■

Note that by equation (101), it follows that

$$(f, s, t) \xrightarrow{(f, s, t, k \diamond k')} (f, s'', t'').$$

This concatenation of quadratic (f, s, t) 2-derivations is clearly associative and it has units; the quadratic (f, s, t) -derivation such that $k(g) = 1_L, \forall g \in G'$. The fact we have a groupoid $[f, f']$ follows from the following lemma:

Lemma 46 If $(f, s, t) \xrightarrow{(f, s, t, k)} (f, s', t')$, then the map $\bar{k}: G' \rightarrow L$ such that $\bar{k}(g) = k(g)^{-1}$ for each $g \in G'$ is a quadratic (f, s', t') 2-derivation.

Proof. By equation (99), and since $(\delta: L \rightarrow E, \triangleright')$ is a crossed module, we have, for each $g, h \in G'$:

$$\begin{aligned}\bar{k}(gh) &\doteq k(gh)^{-1} = \left((s(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright k(g))) k(h) \right)^{-1} \\ &= \left(k(h) ((\delta(k(h))^{-1} s(h)^{-1}) \triangleright' (\phi(h)^{-1} \triangleright k(g))) \right)^{-1} \\ &= (s'(h)^{-1} \triangleright' (\phi(h)^{-1} \triangleright \bar{k}(g))) \bar{k}(h).\end{aligned}$$

■

Given two 2-crossed modules \mathcal{A}' and \mathcal{A} we have a groupoid $[\mathcal{A}', \mathcal{A}]_2$, whose objects are arbitrary 2-crossed module homotopies $f \xrightarrow{(f,s,t)} f'$, where $f, f': \mathcal{A} \rightarrow \mathcal{A}'$, the morphisms being the 2-crossed module 2-fold homotopies. Suppose that \mathcal{A}' is free up to order one, with a chosen basis. To define a 2-groupoid $\text{HOM}_B(\mathcal{A}', \mathcal{A})$, we now need compatible left and right actions of the groupoid $[\mathcal{A}', \mathcal{A}]_1^B$ on $[\mathcal{A}', \mathcal{A}]_2$ (see theorem 43); in other words we need whiskering operators.

3.4.2 Right whiskering 2-fold homotopies by 1-fold homotopies (in the free up to order one case).

Let $\mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \})$ and $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be 2-crossed modules. We now go back to assuming F to be a free group over the chosen basis B .

Let $f, f': \mathcal{A}' \rightarrow \mathcal{A}$ be 2-crossed module maps. Suppose that we have two homotopies (f, s, t) and (f, s', t') connecting f and f' . Suppose that we have a quadratic (f, s, t) 2-derivation $k: F \rightarrow L$ connecting (s, t) and (s', t') , thus $(f, s, t) \xrightarrow{(f,s,t,k)} (f, s', t')$. Since the latter will be a 2-morphism in $\text{HOM}_B(\mathcal{A}', \mathcal{A})$ we now represent it as:

$$\begin{array}{ccc} & (f, s', t') & \\ & \uparrow (f, s, t, k) & \\ f & \xrightarrow{\quad} & f' \\ & \downarrow (f, s, t) & \end{array}$$

Let $f'': \mathcal{A}' \rightarrow \mathcal{A}$ be another 2-crossed module map. Suppose we also have a homotopy

$$f' = (\mu', \psi', \phi') \xrightarrow{(f', s'', t'')} f'' = (\mu'', \psi'', \phi''),$$

so what we have diagrammatically is:

$$\begin{array}{ccccc} & (f, s', t') & & & \\ & \uparrow (f, s, t, k) & & (f', s'', t'') & \\ f & \xrightarrow{\quad} & f' & \xrightarrow{\quad} & f'' \\ & \downarrow (f, s, t) & & & \end{array}$$

Let us define the whiskering:

$$(f, s, t, k) \otimes (f', s'', t'') = (f, s \otimes s'', t \otimes t'', k \otimes s''),$$

such that $k \otimes s''$ connects $(s \otimes s'', t \otimes t'')$ and $(s' \otimes s'', t' \otimes t'')$; diagrammatically:

$$\begin{array}{ccc} & (f, s' \otimes s'', t' \otimes t'') & \\ & \uparrow (f, s, t, k) \otimes (f', s'', t'') & \\ f & \xrightarrow{\quad} & f'' \\ & \downarrow (f, s \otimes s'', t \otimes t'') & \end{array}$$

By definition, $k \otimes s''$ is the unique quadratic $(f, s \otimes s'', t' \otimes t'')$ 2-derivation $F \rightarrow L$, which on the chosen basis B of F has the form:

$$(k \otimes s'')(b) = s''(b)^{-1} \triangleright' k(b).$$

Then we have for each $g \in F$:

$$(s \otimes s'')(g) \delta((k \otimes s'')(g)) = (s' \otimes s'')(g); \quad (112)$$

c.f equation (101). This is because, on the free generators $b \in F$, we have:

$$(s \otimes s'')(b) \delta((k \otimes s'')(b)) = s(b) s''(b) \delta(s''(b)^{-1} \triangleright' k(b)) = s(b) \delta(k(b)) s''(b) = s'(b) s''(b) = (s' \otimes s'')(b).$$

Lemma 47 *The following holds for each $e \in E'$ (c.f. equation (101)):*

$$(k \otimes s'')(\partial(e))^{-1} (t \otimes t'')(e) = (t' \otimes t'')(e). \quad (113)$$

Proof. We freely use the notation of 2.5.2. We have, for each $e \in E'$:

$$(k \otimes s'')(\partial(e))^{-1} (t \otimes t'')(e) = (k \otimes s'')(\partial(e))^{-1} \omega^{(s, s'')}(\partial(e)) s''(\partial(e))^{-1} \triangleright' t(e) t''(e),$$

whereas

$$\begin{aligned} (t' \otimes t'')(e) &= \omega^{(s', s'')}(\partial(e)) s''(\partial(e))^{-1} \triangleright' t'(e) t''(e) \\ &= \omega^{(s \delta(k), s'')}(\partial(e)) s''(\partial(e))^{-1} \triangleright' ((k \circ \partial(e))^{-1} t(e)) t''(e). \end{aligned}$$

Therefore (113) is equivalent to:

$$(k \otimes s'')(\partial(e))^{-1} \omega^{(s, s'')}(\partial(e)) = \omega^{(s \delta(k), s'')}(\partial(e)) s''(\partial(e))^{-1} \triangleright' ((k \circ \partial(e))^{-1} t(e)), \text{ for each } e \in E'.$$

Let us then prove that for any $g \in F$ we have:

$$(k \otimes s'')(g)^{-1} \omega^{(s, s'')}(\partial(e)) = \omega^{(s', s'')}(\partial(e)) s''(\partial(e))^{-1} \triangleright' k(g)^{-1}. \quad (114)$$

Consider the unique map:

$$\begin{aligned} K: F &\rightarrow ((G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L))) \ltimes_{\Delta} (\{1\} \ltimes_{*} (\{1\} \ltimes_{\triangleright'} \{1\}) \ltimes_{*} ((\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L) \ltimes_{\bullet} (\{1\} \ltimes_{\text{ad}} L))) \\ &= \text{Gr}_0(\text{Comp}(\mathcal{A})) \ltimes_{\Delta} \overline{\text{Gr}_1(\text{Comp}(\mathcal{A}))}, \end{aligned} \quad (115)$$

which on the free basis B of F is:

$$K(b) = (\phi(b), s(b), 1, s''(b), 1, 1, 1, 1, \delta(s''(b)^{-1} \triangleright' k(b)), s''(b)^{-1} \triangleright' k(b)^{-1}, 1, 1).$$

By using the morphisms in (89), (87) and (88), 3.3.1, lemma 36 and remark 35 (in 3.4.3 we will make a similar, more difficult, calculation) we have, for each $g \in F$:

$$K(g) = (\phi(g), s(g), 1, s''(g) \delta(\omega^{(s, s'')}(g))^{-1}, \omega^{(s, s'')}(g), 1, 1, 1, 1, \delta(k \otimes s'')(g), (k \otimes s'')(g)^{-1}, 1, 1).$$

Consider the unique map:

$$\begin{aligned} K': F &\rightarrow ((G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L))) \ltimes_{\Delta} (\{1\} \ltimes_{*} (\{1\} \ltimes_{\triangleright'} \{1\}) \ltimes_{*} ((\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L) \ltimes_{\bullet} (\{1\} \ltimes_{\text{ad}} L))) \\ &= \text{Gr}_0(\text{Comp}(\mathcal{A})) \ltimes_{\Delta} \overline{\text{Gr}_1(\text{Comp}(\mathcal{A}))}, \end{aligned} \quad (116)$$

which on the free basis B of F is:

$$K'(b) = (\phi(b), s(b), 1, \delta(k(b)), k(b)^{-1}, 1, 1, 1, 1, s''(b), 1, 1, 1).$$

By using the morphisms in (89), (87) and (88), 3.3.1, equations (86) and (35), lemma 36 and remark 35, we have, for each $g \in F$:

$$K'(g) = (\phi(g), s(g), 1, \delta(k(g)), k(g)^{-1}, 1, 1, 1, 1, s''(g) \delta(\omega^{(s', s'')}(g))^{-1}, \omega^{(s', s'')}(g), 1, 1).$$

By composing K and K' with the group morphism Z of (108) yields two group morphisms $F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_{*} (E \ltimes_{\triangleright'} L)) = \text{Gr}_0(\text{Comp}(\mathcal{A}))$, namely:

$$g \mapsto (\phi(g), s(g), 1, s''(g) \delta(\omega^{(s, s'')}(g))^{-1} \delta(k \otimes s'')(g), (k \otimes s'')(g)^{-1} \omega^{(s, s'')}(g))$$

and

$$g \mapsto (\phi(g), s(g), 1, \delta(k(g)) s''(g) \delta(\omega^{(s', s'')}(g))^{-1}, \omega^{(s', s'')}(g) s''(g)^{-1} \triangleright' k(g)^{-1}).$$

(It is an instructive exercise to check that the last two equations are coherent with (112) and (104).) Since these agree on the chosen basis B of F , they coincide, thus equation (114) follows. This finishes the proof of the lemma ■

Therefore, by (112) and (113), we have that (by equation (101)):

$$(f, s \otimes s'', t \otimes t'') \xrightarrow{(f, s \otimes s'', t \otimes t'', k \otimes s'')} (f, s' \otimes s'', t' \otimes t'').$$

Lemma 48 (Functoriality of the right whiskering) *Let $f, f', f'': \mathcal{A}' \rightarrow \mathcal{A}$ be 2-crossed module maps. Suppose that we are given homotopies (f, s, t) , (f, s', t') and (f, s'', t'') , connecting f and f' , as well as 2-fold homotopies $(f, s, t) \xrightarrow{(f, s, t, k)} (f, s', t')$ and $(f, s', t') \xrightarrow{(f, s', t', k')} (f, s'', t'')$. Suppose we are also given a homotopy $f' \xrightarrow{(f', u, v)} f''$. Diagrammatically we have:*

$$\begin{array}{ccc} & (f, s'', t'') & \\ & \curvearrowright & \\ f & \xrightarrow{(f, s', t')} & f' \\ & \curvearrowleft & \\ & (f, s, t) & \end{array} \xrightarrow{(f', u, v)} f''.$$

Then:

$$((f, s, t, k) \diamond (f, s', t', k')) \otimes (f', u, v) = ((f, s, t, k) \otimes (f', u, v)) \diamond ((f, s', t', k') \otimes (f', u, v)).$$

Proof. In the left-hand-side, since $(f, s, t, k) \diamond (f, s', t', k') = (f, s, t, kk')$, the underlying quadratic $(f, s \otimes u, t \otimes v)$ 2-derivation $(kk') \otimes u$ is the unique quadratic $(f, s \otimes u, t \otimes v)$ 2-derivation $F \rightarrow L$ which on the chosen basis B of F is $b \mapsto u(b)^{-1} \triangleright' (k(b)k'(b))$. On the right hand side we have the quadratic $(f, s \otimes u, t \otimes v)$ 2-derivation $F \rightarrow L$, which is the product of $k \otimes u$ and $k' \otimes u$. On the chosen basis B of F it takes the form:

$$b \mapsto (u(b)^{-1} \triangleright' k(b)) (u(b)^{-1} \triangleright' k'(b)) = u(b)^{-1} \triangleright' (k(b)k'(b)).$$

Therefore:

$$(kk') \otimes u = (k \otimes u) (k' \otimes u),$$

since the same is true in a free basis of F . ■

Analogously:

Lemma 49 *Suppose we have a 2-fold (f, s, t) homotopy $(f, s, t) \xrightarrow{(f, s, t, k)} (f, s', t')$. Consider also a chain of homotopies: $f' \xrightarrow{(f', u, v)} f'' \xrightarrow{(f'', u', v')} f'''$; diagrammatically:*

$$\begin{array}{ccccc} & (f, s', t') & & & \\ & \curvearrowright & & & \\ f & \xrightarrow{(f, s, t, k)} & f' & \xrightarrow{(f', u, v)} & f'' \xrightarrow{(f'', u', v')} f''' \\ & \curvearrowleft & & & \\ & (f, s, t) & & & \end{array}$$

Then:

$$(f, s, t, k) \otimes (u \otimes u', v \otimes v') = ((f, s, t, k) \otimes (u, v)) \otimes (u', v').$$

Proof. In the chosen basis B of F , the underlying quadratic $(f, s \otimes u \otimes u', t \otimes v \otimes v')$ 2-derivation in the left-hand-side is

$$b \mapsto (u \otimes u')(b)^{-1} \triangleright' k(b) = (u(b)u'(b))^{-1} \triangleright' k(b),$$

whereas the underlying quadratic $(f, s \otimes u \otimes u', t \otimes v \otimes v')$ 2-derivation on the right-hand-side is (on free generators of F):

$$b \mapsto u'(b)^{-1} \triangleright' (u(b)^{-1} \triangleright' k(b)),$$

thus these agree on the free basis B of F . ■

Therefore

Proposition 50 Let $\mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \})$ and $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be 2-crossed modules, where F is a free group over the chosen basis B . Whiskering on the right gives a right action (by groupoid morphisms) of the groupoid $[\mathcal{A}', \mathcal{A}]_1^B$ of maps $\mathcal{A} \rightarrow \mathcal{A}'$ and their homotopies, on the groupoid $[\mathcal{A}', \mathcal{A}]_2$ of 2-crossed module homotopies and their 2-fold homotopies.

3.4.3 Left whiskering 2-fold homotopies by 1-fold homotopies (in the free up to order one case)

Let $\mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \})$ and $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be 2-crossed modules, where F is a free group over the chosen basis B . If we have 2-crossed module maps $f, f': \mathcal{A}' \rightarrow \mathcal{A}$, homotopies (f, s, t) and (f, s', t') and a 2-fold homotopy (f, s, t, k) , all fitting into the diagram:

$$\begin{array}{ccc} & (f, s', t') & \\ & \curvearrowright & \\ f & \uparrow (f, s, t, k) & f' \\ & \curvearrowleft & \\ & (f, s, t) & \end{array}$$

and we also have a homotopy

$$f'' = (\mu'', \psi'', \phi'') \xrightarrow{(f'', s'', t'')} f = (\mu, \psi, \phi),$$

so what we have is:

$$\begin{array}{ccccc} & & (f, s', t') & & \\ & & \curvearrowright & & \\ f'' & \xrightarrow{(f'', s'', t'')} & f & \uparrow (f, s, t, k) & f' \\ & & \curvearrowleft & & \\ & & (f, s, t) & & \end{array}$$

let us define the whiskering:

$$(f', s'', t'') \otimes (f, s, t, k) = (f, s'' \otimes s, t'' \otimes t, s'' \otimes k),$$

such that we have:

$$\begin{array}{ccc} & (f'', s'' \otimes s', t'' \otimes t'') & \\ & \curvearrowright & \\ f'' & \uparrow (f'', s'', t'') \otimes (f, s, t, k) & f' \\ & \curvearrowleft & \\ & (f'', s'' \otimes s, t'' \otimes t) & \end{array}$$

By definition, $s'' \otimes k$ is the unique quadratic $(f'', s'' \otimes s, t'' \otimes t)$ 2-derivation which on the chosen basis B of F has the form:

$$(s'' \otimes k)(b) = k(b).$$

Then we have for each $g \in F$:

$$(s'' \otimes s)(g) \delta((s'' \otimes k)(g)) = (s' \otimes s'')(g); \quad (117)$$

c.f. equation (101). This is because on the free generators $b \in F$ we have:

$$(s'' \otimes s)(b) \delta((s'' \otimes k)(b)) = s''(b) s(b) \delta(k(b)) = s''(b) s'(b) = (s'' \otimes s')(b).$$

Lemma 51 We have, for each $e \in E'$:

$$(s'' \otimes k)(\partial(e))^{-1} (t'' \otimes t)(e) = (t'' \otimes t')(e), \quad (118)$$

c.f. equation (101).

Proof. Equation 118 follows if we prove that for each $g \in F$ we have:

$$(s'' \otimes k)(g)^{-1} \omega^{(s'', s)}(g) = \omega^{(s'', s')}(g) k(g)^{-1}. \quad (119)$$

We freely use the notation of 2.5.2. Consider the unique map:

$$K: F \rightarrow ((G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))) \ltimes_{\Delta} (\{1\} \ltimes_* (\{1\} \ltimes_{\triangleright'} \{1\}) \ltimes_* ((\{1\} \ltimes_* (E \ltimes_{\triangleright'} L) \ltimes_{\bullet'} (\{1\} \ltimes_{\text{ad}} L))) \\ = \text{Gr}_0(\text{Comp}(\mathcal{A})) \ltimes_{\Delta} \overline{\text{Gr}_1(\text{Comp}(\mathcal{A}))},$$

which on the free basis B of F is:

$$K(b) = (\phi''(b), s''(b), 1, s(b), 1, 1, 1, 1, \delta(k(b)), 1, 1, k(b)^{-1}).$$

This map has the following form for each $g \in F$:

$$g \xrightarrow{K} (\phi''(g), s''(g), 1, s(g)\delta(\omega^{(s'',s)})(g)^{-1}, \omega^{(s'',s)}(g), 1, 1, 1, \delta(s'' \otimes k(g)), (s'' \otimes k(g))^{-1}k(g), 1, k(g)^{-1}). \quad (120)$$

Let us this time give full details: The underlying map $F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))$ must be $g \mapsto (\phi''(g), s''(g), 1, s(g)\delta(\omega^{(s'',s)})(g)^{-1}, \omega^{(s'',s)}(g))$ by the discussion in 3.3.1. Therefore

$$K(g) = (\phi''(g), s''(g), 1, s(g)\delta(\omega^{(s'',s)})(g)^{-1}, \omega^{(s'',s)}(g), 1, 1, 1, A(g), B(g), 1, C(g)).$$

Composing K with the morphism in (87) yields a group morphism $F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))$ of the form:

$$g \mapsto (\phi''(g), (s'' \otimes s)(g), 1, A(g), B(g)C(g)),$$

which on generators is

$$b \mapsto (\phi''(b), s''(b)s(b), 1, \delta(k(b)), k(b)^{-1}).$$

By remark 35, this morphism must therefore be of the form:

$$g \mapsto (\phi''(g), (s'' \otimes s)(g), 1, \delta(s'' \otimes k)(g), (s'' \otimes k)(g)^{-1}).$$

Composing K with the morphism in (88) yields another group morphism $F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))$ of the form:

$$g \in F \mapsto (\phi(g), s(g), 1, A(g)\delta(B(g)), C(g))$$

which on generators is:

$$b \mapsto (\phi(b), s(b), 1, \delta(k(b)), k(b)^{-1}).$$

By remark 35, this morphism must therefore be of the form:

$$g \in F \mapsto (\phi(g), s(g), 1, \delta(k(g)), k(g)^{-1}).$$

Putting all together yields the form (120) for K .

By composing K with the morphism in (108) yields a group morphism $F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))$ with the form:

$$g \mapsto (\phi''(g), s''(g), 1, s(g)\delta(\omega^{(s'',s)})(g)^{-1}\delta(s'' \otimes k(g)), (s'' \otimes k(g))^{-1}\omega^{(s'',s)}(g)k(g))$$

By lemma 36, we have another group morphism $F \rightarrow (G \ltimes_{\triangleright} E) \ltimes_{\bullet} (\{1\} \ltimes_* (E \ltimes_{\triangleright'} L))$ with the form:

$$g \mapsto (\phi''(g), s''(g), 1, s'(g)\delta(\omega^{(s'',s')})(g)^{-1}, \omega^{(s'',s')}(g))$$

Since these morphisms agree on a basis of F , they coincide, from which (119), thus (118), follows. It is an instructive exercise to check that these calculations are coherent with (117) and (104). ■

From equation (101), by (117) and (118) we therefore have:

$$(f'', s'' \otimes s, t'' \otimes t) \xrightarrow{(f'', s'' \otimes s, t'' \otimes t, s'' \otimes k)} (f, s'' \otimes s', t'' \otimes t').$$

As before we can easily prove that

Proposition 52 *Whiskering on the left gives a left action (by groupoid morphisms) of the groupoid $[\mathcal{A}', \mathcal{A}]_1^B$ of maps $\mathcal{A} \rightarrow \mathcal{A}'$ and their homotopies on the groupoid $[\mathcal{A}', \mathcal{A}]_2$ of 2-crossed module homotopies and their 2-fold homotopies.*

Moreover:

Proposition 53 *Whiskering on the right commutes with whiskering on the left.*

If \mathcal{A}' is free up to order one, with a chosen basis, we have therefore constructed a sesquigroupoid [44] $\text{HOM}_B(\mathcal{A}', \mathcal{A})$ with objects being the 2-crossed module maps $f: \mathcal{A}' \rightarrow \mathcal{A}$, and the morphisms and 2-morphisms being homotopies and 2-fold homotopies. To prove that $\text{HOM}_B(\mathcal{A}', \mathcal{A})$ is a 2-groupoid we now need to prove that it satisfies the interchange law.

3.4.4 The interchange law

Let $\mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \})$ and $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ be 2-crossed modules, where F is a free group over the chosen basis B . Suppose we have the following diagram of 2-crossed module maps $f, f', f'': \mathcal{A}' \rightarrow \mathcal{A}$, homotopies and 2-fold homotopies:

$$\begin{array}{ccccc} & \xrightarrow{(f, s', t')} & & \xrightarrow{(f', u', v')} & \\ f & \xRightarrow{\uparrow (f, s, t, k)} & f' & \xRightarrow{\uparrow (f, u, v, k')} & f'' \\ & \xleftarrow{(f, s, t)} & & \xleftarrow{(f', u, v)} & \end{array}$$

Let us prove the interchange law:

$$((f, s, t, k) \otimes (f', u, v)) \diamond ((f, s', t') \otimes (f, u, v, k')) = ((f, s, t) \otimes (f, u, v, k')) \diamond ((f, s, t, k) \otimes (f', u', v')).$$

In the left hand side we have the quadratic $(f, s \otimes u, t \otimes v)$ 2-derivation, which on the basis B of F is:

$$b \mapsto (u^{-1}(b) \triangleright' k(b)) k'(b).$$

In the right hand side we have the $(f, s \otimes u, t \otimes v)$ 2-derivation, which on the chosen basis B of F is:

$$b \mapsto k'(b) u'(b)^{-1} \triangleright' k(b) = k'(b) (\delta(k'(b)^{-1})u(b)^{-1}) \triangleright' k(b) = (u^{-1}(b) \triangleright' k(b)) k'(b),$$

where we have used (101) and the crossed module rules, recalling that $(\delta: L \rightarrow E, \triangleright')$ is a crossed module.

We therefore proved that:

Theorem 54 (Mapping space 2-groupoid) *Given two 2-crossed modules $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ and $\mathcal{A}' = (L' \xrightarrow{\delta} E' \xrightarrow{\partial} F, \triangleright, \{, \})$, with F free up to order 1, with a chosen basis B of F , there exists a 2-groupoid:*

$$\text{HOM}_B(\mathcal{A}', \mathcal{A})$$

of 2-crossed module maps, 1-fold homotopies between 2-crossed module maps, and 2-fold homotopies between 1-fold homotopies.

We note that $\text{HOM}_B(\mathcal{A}', \mathcal{A})$ explicitly depends on the chosen basis B of F .

4 Lax homotopy of 2-crossed modules

4.1 Definition of $Q^1(\mathcal{A})$ and lax homotopy

Consider a 2-crossed module of groups:

$$\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \}). \quad (121)$$

We will consider a very natural partial resolution $Q^1(\mathcal{A})$ of it, which is free up to order one, with a chosen basis, together with a surjective projection $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$, defining isomorphisms at the level of 2-crossed module homotopy groups. It is proven in [25] that Q^1 (clearly functorial by its construction) is a part of a comonad. We will then use $Q^1(\mathcal{A})$ to define lax homotopy of (strict) 2-crossed module maps.

4.1.1 Construction of $Q^1(\mathcal{A})$ and abstract definition of Q^1 -lax homotopy

Let G be a group. The free group on the underlying set of G is denoted by $\mathcal{F}^{\text{group}}(G)$. The inclusion (set) map $G \rightarrow \mathcal{F}^{\text{group}}(G)$ is denoted by $g \in G \mapsto [g] \in \mathcal{F}^{\text{group}}(G)$. The projection (group) map sending $[g] \in \mathcal{F}^{\text{group}}(G)$ to $g \in G$ is denoted by $p: \mathcal{F}^{\text{group}}(G) \rightarrow G$. Note that we do not take $[1]$, where 1 is the identity of G , to be the identity of $\mathcal{F}^{\text{group}}(G)$, the latter being the empty word, denoted by \emptyset .

For the 2-crossed module (121), we put:

$$Q^1(\mathcal{A}) = (L \xrightarrow{\delta'} E_{\partial} \times p\mathcal{F}^{\text{group}}(G) \xrightarrow{\partial'} \mathcal{F}^{\text{group}}(G), \triangleright, \{, \}),$$

where of course:

$$E_{\partial} \times p\mathcal{F}^{\text{group}}(G) = \{(e, u) \in E \times \mathcal{F}^{\text{group}}(G) : \partial(e) = p(u)\}.$$

Moreover $\partial'(e, u) = u$, and, for all $u \in \mathcal{F}^{\text{group}}(G)$, we put $u \triangleright (e, u') = (p(u) \triangleright e, uu'u^{-1})$. Clearly the Peiffer pairing is $\langle (e, u), (e', u') \rangle = (\langle e, e' \rangle, \emptyset)$. We also put $\delta'(k) = (\delta(k), \emptyset)$ and also $u \triangleright k = p(u) \triangleright k$, for all $k \in L$ and $u \in \mathcal{F}^{\text{group}}(G)$. It is immediate that with the Peiffer lifting:

$$\{(e, u), (e', u')\} = \{e, e'\},$$

this defines a 2-crossed module of groups (a very similar construction appears in [1]). Also, $Q^1(\mathcal{A})$ is free up to order one, and we choose the free basis $[G] = \{[g], g \in G\}$ of $\mathcal{F}^{\text{group}}(G)$.

There is a projection $\text{proj} = (r, q, p): Q^1(\mathcal{A}) \rightarrow \mathcal{A}$ which, rather clearly, yields isomorphisms at the level of 2-crossed module homotopy groups (the homology groups of the underlying complexes). It has the form:

$$\begin{array}{ccccc} L & \xrightarrow{\delta'} & E_{\partial} \times p\mathcal{F}^{\text{group}}(G) & \xrightarrow{\partial'} & \mathcal{F}^{\text{group}}(G) \\ \text{proj} = & r \downarrow & q \downarrow & & p \downarrow \\ L & \xrightarrow{\delta} & E & \xrightarrow{\partial} & G. \end{array} \quad (122)$$

where $r = \text{id}$ and $q(e, u) = e$, for each $(e, u) \in E_{\partial} \times p\mathcal{F}^{\text{group}}(G)$.

Consider 2-crossed modules $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ and $\mathcal{A}' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} G', \triangleright, \{, \})$. If we have a 2-crossed module morphism $f: \mathcal{A} \rightarrow \mathcal{A}'$ then $f \circ \text{proj}$ is a 2-crossed module morphism $Q^1(\mathcal{A}) \rightarrow \mathcal{A}'$. This yields an injective map $\text{proj}^*: \text{hom}(\mathcal{A}, \mathcal{A}') \rightarrow \text{hom}(Q^1(\mathcal{A}), \mathcal{A}')$, since $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$ is surjective. Here $\text{hom}(\mathcal{A}, \mathcal{A}')$ denotes the set of 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$.

Definition 55 (Lax mapping space) A morphism $Q^1(\mathcal{A}) \rightarrow \mathcal{A}'$ is said to be a strict map $\mathcal{A} \rightarrow \mathcal{A}'$ if it factors (uniquely) through $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$. The lax mapping space 2-groupoid:

$$\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$$

is the full sub-2-groupoid of $\text{HOM}_{[G]}(Q^1(\mathcal{A}), \mathcal{A}')$, theorem 54, with objects the strict maps $f: \mathcal{A} \rightarrow \mathcal{A}'$, each uniquely identified with $f \circ \text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}'$. The objects of $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$ are therefore in one-to-one correspondence with 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$, and we call the 1- and 2-morphisms of $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$ “lax homotopies” and “lax 2-fold homotopies”.

4.1.2 The structure of $Q^1(\mathcal{A})$

We now want to completely unpack the definition of $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$, definition 55. To unravel the structure of $Q^1(\mathcal{A})$, we prove an auxiliary lemma, describing the kernel of the projection map $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$ in (122). To characterize this kernel it suffices to elucidate the kernel $\ker(p)$ of the obvious projection $p: \mathcal{F}^{\text{group}}(G) \rightarrow G$. Then the kernel of the projection map $\text{proj}: Q^1(\mathcal{A}) \rightarrow \mathcal{A}$ is the 2-crossed module:

$$\ker(\text{proj}) = \{1 \rightarrow \{1_E\} \times \ker(p) \rightarrow \ker(p)\},$$

with action by conjugation and trivial Peiffer lifting.

For details on the definition of free crossed modules and free pre-crossed modules (possibly with ulterior relations), we refer to [20, 10]. Let G be a group. Given $g, h \in G$ put:

$$[g, h] = [gh]^{-1} [g] [h] \in \ker(p) \subset \mathcal{F}^{\text{group}}(G). \quad (123)$$

Note that we always have:

$$[gh, i] [i]^{-1} [g, h] [i] = [g, hi] [h, i], \text{ where } g, h, i \in G. \quad (124)$$

Also if $g, h \in G$:

$$\begin{aligned} [g] [h] [g, h]^{-1} &= [gh], \\ [1] &= [1, 1], \\ [g^{-1}] &= [g]^{-1} [1] [g, g^{-1}]. \end{aligned} \quad (125)$$

Moreover:

$$\begin{aligned} [ghg^{-1}] &= [g] [hg^{-1}] [g, hg^{-1}]^{-1} = [g] [h] [g^{-1}] [h, g^{-1}]^{-1} [g, hg^{-1}]^{-1} \\ &= [g] [h] [g]^{-1} [1] [g, g^{-1}] [h, g^{-1}]^{-1} [g, hg^{-1}]^{-1}. \end{aligned} \quad (126)$$

Lemma 56 *The inclusion map $\iota: \ker(p) \rightarrow \mathcal{F}^{\text{group}}(G)$, together with the action \triangleright of $\mathcal{F}^{\text{group}}(G)$ on $\ker(p) \subset \mathcal{F}^{\text{group}}(G)$ by conjugation, is isomorphic to the crossed module, over $\mathcal{F}^{\text{group}}(G)$, formally generated by the elements (g, h) , where $g, h \in G$, with:*

$$\iota(g, h) = [g, h], \text{ for each } g, h \in G,$$

modulo the relations:

$$(gh, i) [i]^{-1} \triangleright (g, h) = (g, hi) (h, i), \text{ where } g, h, i \in G. \quad (127)$$

In particular, $\ker(p) \rightarrow \mathcal{F}^{\text{group}}(G)$ is isomorphic to the pre-crossed module, formally generated by the symbols (g, h) , for all $g, h \in G$, with $\iota(g, h) = [g, h]$, modulo the relations (127), as well as the following relations (where $g, g', h, h' \in G$ and $k, k' \in \mathcal{F}^{\text{group}}(G)$), enforcing the second Peiffer condition in definition 1:

$$(\iota(k \triangleright (g, h))) \triangleright (k' \triangleright (g', h')) = (k \triangleright (g, h)) (k' \triangleright (g', h')) (k \triangleright (g, h))^{-1}.$$

(It suffices to consider the case $k = \emptyset$.)

Also, we have that $\ker(p) \subset \mathcal{F}^{\text{group}}(G)$, as a group, is generated by all conjugates (under $\mathcal{F}^{\text{group}}(G)$) of elements $[g, h]$, where $g, h \in g$, and the relations (127) are the only relations between these.

We will give a topological proof of this lemma. Recall [10] that, if (X, Y) is a pair of path-connected spaces, then the boundary map $\pi_2(X, Y) \rightarrow \pi_1(Y)$, together with the standard action of π_1 on π_2 , defines a crossed module $\Pi_2(X, Y)$, a result due to Whitehead. If X is obtained from Y by attaching 2-cells, then $\Pi_2(X, Y)$ is the free crossed module on the attaching maps of the 2-cells of X in $\pi_1(Y)$, a fact usually known as Whitehead theorem [46, 47, 48]. If X is a CW-complex, then X^i denotes the i -skeleton of X .

Proof. Let K be the simplicial set which is the nerve of G , thus the geometric realisation of K is the usual classifying space of G ; see for example [10]. We thus have a unique 0-simplex, the 1-simplices of K are in one-to-one correspondence with elements of G , and we denote these by $[g]$, where $g \in G$. The 2-simplices of K are in one-to-one correspondence with pairs (g, h) , where $g, h \in G$, being:

$$\partial_0(g, h) = [h] \quad \partial_1(g, h) = [gh] \quad \partial_2(g, h) = [g].$$

The 3-simplices of K are triples (g, h, i) of elements of G , being:

$$\partial_0(g, h, i) = (h, i) \quad \partial_1(g, h, i) = (gh, i) \quad \partial_2(g, h, i) = (g, hi) \quad \partial_3(g, h, i) = (g, h).$$

Let us consider the fat geometric realization X of K (forgetting about the degeneracy maps of K , thus looking at K merely as being a Δ -complex, [30].) It is well known that the fat and standard geometric realisations of a simplicial set are homotopy equivalent (see for example [11]), thus X is an aspherical CW-complex with $\pi_1(X) \cong G$.

Consider the exact sequence $\{1\} \rightarrow \pi_2(X, X^1) \rightarrow \pi_1(X^1) \rightarrow \pi_1(X) \cong G$. These are the final groups of the long homotopy exact sequence of the pair (X, X^1) , where X^1 is the 1-skeleton of X . Then $\pi_1(X^1) \rightarrow \pi_1(X)$ is exactly the map $p: \mathcal{F}^{\text{group}}(G) \rightarrow G$, so we only need to determine $\pi_2(X, X^1)$. The crossed module $(\pi_2(X, X^1) \rightarrow \pi_1(X^1)) = \Pi_2(X, X^1)$ is a quotient of the crossed module $(\pi_2(X^2, X^1) \rightarrow \pi_1(X^1)) = \Pi_2(X^2, X^1)$, which, by Whitehead theorem, is the free crossed module on the boundary maps of the 2-cells of X in $\pi_1(X^1)$. Thus $\pi_2(X^2, X^1)$ is the principal group of the pre-crossed module, over $\pi_1(X^1)$, generated by all pairs (g, h) where $g, h \in G$, with $\iota(g, h) = [g, h]$, modulo the relations (for $g, g', h, h' \in G$ and $k, k' \in \mathcal{F}^{\text{group}}(G)$), enforcing the second Peiffer condition in definition 1:

$$(\iota(k \triangleright (g, h))) \triangleright (k' \triangleright (g', h')) = (k \triangleright (g, h)) (k' \triangleright (g', h')) (k \triangleright (g, h))^{-1}.$$

To obtain $\pi_2(X, X^1)$ from $\pi_2(X^2, X^1)$, we now need to add one extra relation for each 3-cell of X (see [20]), yielding that we should have:

$$(gh, i) [i]^{-1} \triangleright (g, h) = (g, hi) (h, i), \text{ where } g, h, i \in G,$$

which arise from all 3-simplices of K . ■

4.1.3 Explicit form of a lax homotopy between two strict 2-crossed module maps

We freely use the notation of subsection 3.1 and 4.1.1. Consider 2-crossed modules $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ and $\mathcal{A}' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} G', \triangleright, \{, \})$. Recall the construction of $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$, definition 55, from $Q^1(\mathcal{A}) = (L \xrightarrow{\delta'} E_{\partial} \times p\mathcal{F}^{\text{group}}(G) \xrightarrow{\partial'} \mathcal{F}^{\text{group}}(G), \triangleright, \{, \})$. Let $f_1 = (\mu_1, \psi_1, \phi_1)$ and $f_2 = (\mu_2, \psi_2, \phi_2)$ be 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$. Consider a lax homotopy connecting f_1 and f_2 , namely:

$$(\mu'_1, \psi'_1, \phi'_1) = f'_1 \doteq (f_1 \circ \text{proj}) \xrightarrow{(f_1 \circ \text{proj}, s, t)} (f_2 \circ \text{proj}) \doteq f'_2 = (\mu'_2, \psi'_2, \phi'_2),$$

thus (s, t) is a quadratic f'_1 -derivation connecting $f'_1 = (f_1 \circ \text{proj})$ and $f'_2 = (f_2 \circ \text{proj})$. Therefore $\phi'_1([g]) = \phi_1(g)$ and $\phi'_2([g]) = \phi_2(g)$, for each $g \in G$. Also $\phi'_1([g, h]) = \phi'_2([g, h]) = 1_G$ and $\psi'_1(1, [g, h]) = \psi'_2(1, [g, h]) = 1_E$, for each $g, h \in G$. Following the notation of lemma 56, let us put $(g, h) = (1, [g, h]) \in E_{\partial} \times p\mathcal{F}^{\text{group}}(G)$, thus $\partial'(g, h) = [g, h] = [gh]^{-1}[g][h]$.

Let us look at the associated group map (lemma 28):

$$(\phi'_1, s): \mathcal{F}^{\text{group}}(G) \rightarrow G' \ltimes_{\triangleright} E'.$$

Note the following equation, which will be used several times (we use (90) and $\phi'_1([gh]) = \phi_1(gh)$):

$$\begin{aligned} (s \circ \partial')(g, h) &= s([gh]^{-1}[g][h]) = \phi_1(gh)^{-1} \triangleright s([gh]^{-1}) \phi_1(h)^{-1} \triangleright s([g]) s([h]) \\ &= s([gh])^{-1} \phi_1(h)^{-1} \triangleright s([g]) s([h]). \end{aligned} \quad (128)$$

The fact that $\phi'_2(\ker(p)) = 1$ and $\phi'_1(\ker(p)) = 1$ tells us that for each $g, h \in G$:

$$\begin{aligned} 1 &= \phi'_2([g, h]) = \phi'_2([gh]^{-1}[g][h]) = \phi'_1([gh]^{-1}[g][h]) \partial(s([gh]^{-1}[g][h])) = \partial(s([gh]^{-1}[g][h])) \\ &= \partial(s([gh])^{-1} \phi_1(h)^{-1} \triangleright s([g]) s([h])). \end{aligned}$$

Thus we must have that, for each $g, h \in G$:

$$\partial(s([gh])) = \partial(\phi_1(h)^{-1} \triangleright s([g]) s([h])). \quad (129)$$

Let us now look at the group map (lemma 28):

$$(\psi'_1, s \circ \partial', t): E_{\partial} \times p\mathcal{F}^{\text{group}}(G) \rightarrow E' \ltimes_* (E' \ltimes_{\triangleright} L').$$

Let

$$\Pi(g, h) = t(g, h), \text{ where } g, h \in G.$$

Noting that $\psi'_1(g, h) = 1$, we have:

$$\begin{aligned} (\psi'_1, s \circ \partial', t)(g, h) &= (1, (s \circ \partial')(g, h), t(g, h)) = (1, s([gh])^{-1} \phi_1(h)^{-1} \triangleright s([g]) s([h]), t(g, h)) \\ &= (1, s([gh])^{-1} \phi_1(h)^{-1} \triangleright s([g]) s([h]), \Pi(g, h)). \end{aligned}$$

The fact that $\psi'_2(g, h) = 1$ tells us that:

$$1 = (s \circ \partial')(g, h) \delta(\Pi(g, h)) = s([gh])^{-1} \phi_1(h)^{-1} \triangleright s([g]) s([h]) \delta(\Pi(g, h)).$$

Therefore, for each $g, h \in G$, we have:

$$s([gh]) = \phi_1(h)^{-1} \triangleright s([g]) s([h]) \delta(\Pi(g, h)), \quad (130)$$

thus also for $g, h \in G$:

$$(\psi'_1, s \circ \partial', t)(g, h) = (1, \delta(\Pi(g, h))^{-1}, \Pi(g, h)), \quad (131)$$

and

$$s([g, h]) = \delta(\Pi(g, h))^{-1}, \quad (132)$$

thus in particular:

$$s([1]) = s([1, 1]) = \delta(\Pi(1, 1))^{-1}. \quad (133)$$

Since $((\psi'_1, s \circ \partial', t), (\phi'_1, s))$ is, by lemma 28, a pre-crossed module map into $(\beta: E' \ltimes_* (E' \ltimes_{\triangleright'} L') \rightarrow G' \ltimes_{\triangleright} E', \bullet)$ we have:

$$\begin{aligned} (\psi'_1, s \circ \partial', t)(1, k \triangleright [g, h]) &= (\psi'_1, s \circ \partial', t)(k \triangleright (g, h)) = (\phi'_1(k), s(k)) \bullet (\psi'_1, s \circ \partial', t)(g, h) \\ &= (\phi_1(p(k)), s(k)) \bullet (1, s([gh])^{-1} \phi_1(h)^{-1} \triangleright s([g]) s([h]), t(g, h)), \end{aligned} \quad (134)$$

where $k \in \mathcal{F}^{\text{group}}(G)$ and $g, h \in G$. The fact that $\psi'_2(k \triangleright (g, h)) = 1$ is equivalent (by (134) and lemmas 27 and 28) to:

$$\phi_1(p(k))\partial(s(k)) \triangleright (s([gh])^{-1} \phi_1(h)^{-1} \triangleright s([g]) s([h]) \delta(\Pi(g, h))) = 1,$$

therefore this fact is implied by (130).

Let us now find necessary and sufficient conditions for

$$k \in \mathcal{F}^{\text{group}}(G) \mapsto (\phi_1(p(k)), s(k)) \in G' \ltimes_{\triangleright} E'$$

and

$$(e, k) \in E_{\partial} \times p\mathcal{F}^{\text{group}}(G) \mapsto (\psi_1(q(e, k)), s(\partial'(e, k)), t(e, k)) \in E' \ltimes_* (E' \ltimes_{\triangleright'} L')$$

to be a pre-crossed module morphism, yielding a lax homotopy between the strict 2-crossed module maps f_1 and f_2 . As far as $(\phi_1(p(k)), s(k))$ is concerned, since $\mathcal{F}^{\text{group}}(G)$ is free on the underlying set of G , there are no conditions on s to add to (129).

Given $(e, k) \in E_{\partial} \times p\mathcal{F}^{\text{group}}(G)$ we have

$$(e, k) = (e, [\partial(e)]) (1, [\partial(e)]^{-1}k),$$

where clearly $[\partial(e)]^{-1}k \in \ker(p)$. By using this and lemma 56, we can easily see that the group $E_{\partial} \times p\mathcal{F}^{\text{group}}(G)$ is the principal group of the $\mathcal{F}^{\text{group}}(G)$ pre-crossed module, formally generated by the symbols $[e] \doteq (e, [\partial(e)])$, with $e \in E$, and $(g, h) \doteq (1, (g, h))$, with $g, h \in G$, modulo the relations (135), below (for $g, h \in G$, $e, f \in E$ and $l \in \mathcal{F}^{\text{group}}(G)$), which (we leave the reader to verify this) do hold in $E_{\partial} \times p\mathcal{F}^{\text{group}}(G)$:

$$\begin{aligned} [e][f] &= [ef] (\partial(e), \partial(f)) \\ g \triangleright [e] &= [g \triangleright e] (g, \partial(e)g^{-1}) (\partial(e), g^{-1}) (g, g^{-1})^{-1} (1, 1)^{-1} \\ (gh, i) [i]^{-1} \triangleright (g, h) &= (g, hi) (h, i) \\ \partial(g, h) \triangleright (l \triangleright (g', h')) &= (g, h) (l \triangleright (g', h')) (g, h)^{-1} \text{ or } ([g, h]l) \triangleright (g', h') = (g, h) (l \triangleright (g', h')) (g, h)^{-1}. \end{aligned} \quad (135)$$

Let $t(a) = t([a])$ and $s(g) = s([g])$, where $g \in G$ and $a \in E$. Note that $\psi'_1([e]) = \psi_1(e)$ and $\psi'_2([e]) = \psi_1(e)$. Then $t: E_{\partial} \times p\mathcal{F}^{\text{group}}(G) \rightarrow L'$ can be specified by $t(a)$ and $t(g, h) = \Pi(g, h)$, which must satisfy relations (135). These translate into (in order of appearance):

$$(\psi_1(a), s(\partial(a)), t(a)) (\psi_1(b), s(\partial(b)), t(b)) = (\psi_1(ab), s(\partial(ab)), t(ab)) (1, \delta(\Pi(\partial(a), \partial(b))^{-1}, \Pi(\partial(a), \partial(b))),$$

which is the same as, for any $a, b \in G$, (by (19)):

$$\begin{aligned} & \left(\psi_1(a)\psi_1(b), \phi_1(\partial(b))^{-1} \triangleright s(a) s(\partial(b)), ((s \circ \partial)(b))^{-1} \triangleright' \{\psi_1(b)^{-1}, s(\partial(a))^{-1}\}^{-1} \right. \\ & \left. \left((\psi_1(b)((s \circ \partial)(b))^{-1} \triangleright' t(a)) t(b) \right) \right) = \left(\psi_1(ab), s(\partial(ab)) \delta(\Pi(\partial(a), \partial(b))^{-1}, \Pi(\partial(a), \partial(b))) t(ab) \right). \end{aligned}$$

At the level of the first two components, equality always hold, given the calculations above. Therefore

$$\Pi(\partial(a), \partial(b)) t(ab) = ((s \circ \partial)(b))^{-1} \triangleright' \left(\{\psi_1(b)^{-1}, s(\partial(a))^{-1}\}^{-1} \psi_1(b)^{-1} \triangleright' t(a) \right) t(b). \quad (136)$$

The second relation translates into (for any $g \in G$ and $a \in E$):

$$\begin{aligned} (\phi(g), s(g)) \bullet (\psi(a), s(\partial(a)), t(a)) &= (\psi_1(g \triangleright a), s(\partial(g \triangleright a)), t(g \triangleright a)) (1, \delta(\Pi(g, \partial(a)g^{-1}))^{-1}, \Pi(g, \partial(a)g^{-1})) \\ & \quad (1, \delta(\Pi(\partial(a), g^{-1}))^{-1}, \Pi(\partial(a), g^{-1})) (1, \delta(\Pi(g, g^{-1})), \Pi(g, g^{-1})^{-1}) (1, \delta(\Pi(1, 1)), \Pi(1, 1)^{-1}) \end{aligned}$$

which gives:

$$\begin{aligned} \phi_1(g) \triangleright \left(s(g) s(\partial(a))^{-1} \triangleright' \left\{ \psi_1(a)^{-1}, s(g)^{-1} \right\}^{-1} \right) \phi_1(g) \triangleright \left\{ s(g), s(\partial(a))^{-1} \psi_1(a)^{-1} \right\} (\phi_1(g) (\partial \circ s)(g)) \triangleright t(a) \\ = \Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(\partial(a), g^{-1}) \Pi(g, \partial(a) g^{-1}) t(g \triangleright a). \end{aligned} \quad (137)$$

By using (32) and (35), the third relation translates into the cocycle type identity (for each $g, h, i \in G$):

$$s(i) \triangleright' (\phi_1(i)^{-1} \triangleright \Pi(g, h)) \Pi(gh, i) = \Pi(h, i) \Pi(g, hi). \quad (138)$$

Let us see that the fourth relation is void in this case. Note that, for $g, h \in G$ and $k \in \mathcal{F}^{\text{group}}(G)$, we have:

$$(\psi'_1, (s \circ \partial'), t)(k \triangleright (g, h)) = (\phi_1(p(k)), s(k)) \bullet (\psi_1, (s \circ \partial'), t)(g, h) = (\phi_1(p(k)), s(k)) \bullet (1, \delta(\Pi(g, h))^{-1}, \Pi(g, h)).$$

Applying $(\psi'_1, (s \circ \partial), t)$ to the left hand side of the last relation of (135) yields (since $\phi_1(p([g, h])) = 1$ and by using (32)):

$$\begin{aligned} (\phi_1(p(l)), s([g, h]l)) \bullet (1, \delta(\Pi(g', h'))^{-1}, \Pi(g', h')) \\ = \phi_1(p_1(l)) \triangleright (1, s([g, h]l) \delta(\Pi(g', h'))^{-1} s([g, h]l)^{-1}, s([g, h]l) \triangleright' \Pi(g', h')). \end{aligned}$$

Applying $(\psi'_1, (s \circ \partial), t)$ to the right-hand-side of the last relation of (135) gives:

$$\begin{aligned} (1, \delta(\Pi(g, h))^{-1}, \Pi(g, h)) (\phi_1(p(l)), s(l)) \bullet (1, \delta(\Pi(g', h'))^{-1}, \Pi(g', h')) (1, \delta(\Pi(g, h))^{-1}, \Pi(g, h))^{-1} \\ = (1, \delta(\Pi(g, h))^{-1}, \Pi(g, h)) \phi_1(p(l)) \triangleright (1, s(l) \delta(\Pi(g', h'))^{-1} s(l)^{-1}, s(l) \triangleright' \Pi(g', h')) (1, \delta(\Pi(g, h))^{-1}, \Pi(g, h))^{-1}. \end{aligned}$$

Therefore both sides applied by $(\psi'_1, (s \circ \partial), t)$ coincide since (we use the fact that $(\delta: L \rightarrow E, \triangleright')$ is a crossed module):

$$\begin{aligned} \Pi(g, h)^{-1} \phi_1(p(l)) \triangleright (s(l) \triangleright' \Pi(g', h')) \Pi(g, h) &= \phi_1(p(l)) \triangleright \left((\delta(\phi_1(p(l))^{-1} \triangleright \Pi(g, h)^{-1}) s(l)) \triangleright' \Pi(g', h') \right) \\ &= \phi_1(p(l)) \triangleright \left((\phi_1(p(l))^{-1} \triangleright s([g, h]) s(l)) \triangleright' \Pi(g', h') \right) \\ &= \phi_1(p(l)) \triangleright \left((s([g, h]l)) \triangleright' \Pi(g', h') \right). \end{aligned}$$

We have (almost) proven:

Theorem 57 Consider 2-crossed modules $\mathcal{A} = \left(L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \} \right)$ and $\mathcal{A}' = \left(L' \xrightarrow{\delta} E' \xrightarrow{\partial} G', \triangleright, \{, \} \right)$. Let $f_1 = (\mu_1, \psi_1, \phi_1)$ and $f_2 = (\mu_2, \psi_2, \phi_2)$ be 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$. A lax homotopy connecting f_1 and f_2 , definition 55, which we write as:

$$f_1 \xrightarrow{(f_1, \hat{s}, \hat{t}, \Pi)} f_2,$$

is given by:

1. A (set) map $\hat{s}: G \rightarrow E'$,
2. A (set) map $\hat{t}: E \rightarrow L'$,
3. A (set) map $\Pi: G \times G \rightarrow L'$.

These are to satisfy, for each $g, h \in G$ and $a, b \in E$, that:

$$\partial(\hat{s}(gh)) = \partial(\phi_1(h)^{-1} \triangleright \hat{s}(g) \hat{s}(h)), \quad (139)$$

$$\hat{s}(gh) = \phi_1(h)^{-1} \triangleright \hat{s}(g) \hat{s}(h) \delta(\Pi(g, h)), \quad (140)$$

$$\Pi(\partial(a), \partial(b)) \hat{t}(ab) = ((\hat{s} \circ \partial)(b))^{-1} \triangleright' \left(\{ \psi_1(b)^{-1}, \hat{s}(\partial(a))^{-1} \}^{-1} \psi_1(b)^{-1} \triangleright' \hat{t}(a) \right) \hat{t}(b), \quad (141)$$

$$\begin{aligned}
\phi_1(g) \triangleright \left(\hat{s}(g) \hat{s}(\partial(a))^{-1} \triangleright' \left\{ \psi_1(a)^{-1}, \hat{s}(g)^{-1} \right\}^{-1} \right) \phi_1(g) \triangleright \left\{ \hat{s}(g), \hat{s}(\partial(a))^{-1} \psi_1(a)^{-1} \right\} (\phi_1(g)(\partial \circ \hat{s})(g)) \triangleright \hat{t}(a) \\
= \Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(\partial(a), g^{-1}) \Pi(g, \partial(a)g^{-1}) \hat{t}(g \triangleright a), \quad (142) \\
\hat{s}(i) \triangleright' (\phi_1(i)^{-1} \triangleright \Pi(g, h)) \Pi(gh, i) = \Pi(h, i) \Pi(g, hi). \quad (143)
\end{aligned}$$

And, moreover, if $g \in G$, $e \in E$ and $l \in L$:

$$\begin{aligned}
\phi_2(g) &= \phi_1(g) \partial(\hat{s}(g)), \\
\psi_2(e) &= \psi_1(e) \hat{s}(\partial(e)) \delta(\hat{t}(e)), \\
\mu_2(l) &= \mu_1(l) \Pi(1, 1)^{-1} \hat{t}(\delta(l)).
\end{aligned} \quad (144)$$

Moreover, the corresponding (strict) homotopy between strict 2-crossed module maps $Q^1(\mathcal{A}) \rightarrow \mathcal{A}'$, namely:

$$f'_1 = f_1 \circ \text{proj} \xrightarrow{(f_1 \circ \text{proj}, s, t)} f_2 \circ \text{proj} = f'_2$$

is given by the $(f_1 \circ \text{proj})$ -quadratic derivation (s, t) , where $s: \mathcal{F}^{\text{group}}(G) \rightarrow E'$ is the unique $(\phi_1 \circ p)$ -derivation (definition 24) extending $\hat{s}: G \rightarrow E'$, and on the group generators of $E_\partial \times p\mathcal{F}^{\text{group}}(G)$ we have that $t(e, [\partial(e)]) = \hat{t}(e)$ and $t(g, h) = \Pi(g, h)$, where $e \in E$ and $g, h \in G$. Recall that $p: \mathcal{F}^{\text{group}}(G) \rightarrow G$ is the obvious projection.

We thus have a lax analogue of the strict homotopy relation treated in subsection 3.1.

Proof. We just need to check the last equation of (144). Note that if $l \in L$:

$$\delta'(l) = (\delta(l), \emptyset) = (\delta(l), [1]) (1, [1]^{-1}) = [\delta(l)] (1, 1)^{-1}.$$

Therefore, by using (93) and (133), and noting $\psi'([1]) = 1_{E'}$:

$$\hat{t}(\delta'(l)) = \hat{s}([1]^{-1})^{-1} \triangleright' \hat{t}(\delta(l)) \hat{t}((1, 1)^{-1}) = \hat{s}([1]^{-1})^{-1} \triangleright' (\hat{t}(\delta(l)) (\hat{t}(1, 1))^{-1}).$$

Thus

$$\mu_2(l) = \mu'_2(l) = \mu_1(l) \hat{t}(\delta'(l)) = \mu_1(l) \hat{s}([1]^{-1})^{-1} \triangleright' (\hat{t}(\delta(l)) \Pi(1, 1)^{-1}).$$

And now note that $\hat{s}([1]^{-1}) = \phi'_1([1]) \triangleright \hat{s}([1])^{-1} = \hat{s}([1])^{-1}$, since $\phi'_1([1]) = 1$, together with (133), and the second Peiffer condition in definition 1. ■

It follows by construction that $f_2 = (\mu_2, \psi_2, \phi_2)$, defined in (144), is a 2-crossed module morphism $\mathcal{A} \rightarrow \mathcal{A}'$, if equations (139) to (143) are satisfied. Let us nevertheless elucidate how this could be proven directly. First note that if $g \in G$ and $e \in E$:

$$\begin{aligned}
\hat{s}(g\partial(e)g^{-1}) &= \phi_1(g\partial(e)g^{-1}) \triangleright \hat{s}(g) \phi(g) \triangleright \hat{s}(\partial(e)) \hat{s}(g^{-1}) \delta(\Pi(\partial(e), g^{-1})) \delta(\Pi(g, \partial(e)g^{-1})) \\
&= \phi_1(g\partial(e)g^{-1}) \triangleright \hat{s}(g) \phi_1(g) \triangleright \hat{s}(\partial(e)) \phi_1(g) \triangleright \hat{s}(g)^{-1} \hat{s}(1) \delta(\Pi(g, g^{-1})^{-1} \delta(\Pi(\partial(e), g^{-1})) \delta(\Pi(g, \partial(e)g^{-1})).
\end{aligned}$$

Let us prove that $\psi_2: E \rightarrow E'$ is G -equivariant. If $e \in E$ and $g \in G$:

$$\begin{aligned}
\psi_2(g \triangleright e) &= \psi_1(g \triangleright e) (\hat{s} \circ \partial)(g \triangleright e) (\delta \circ t)(g \triangleright e) = \phi_1(g) \triangleright \psi_1(e) \phi_1(g\partial(e)g^{-1}) \triangleright \hat{s}(g) \phi_1(g) \triangleright \hat{s}(\partial(e)) \phi_1(g) \triangleright \hat{s}(g)^{-1} \\
&\quad \hat{s}(1) \delta(\Pi(g, g^{-1})^{-1} \delta(\Pi(\partial(e), g^{-1})) \delta(\Pi(g, \partial(e)g^{-1})) \delta(\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(\partial(e), g^{-1}) \Pi(g, \partial(e)g^{-1}))^{-1})^{-1} \\
&\quad \phi_1(g) \triangleright \left(\hat{s}(g) \hat{s}(\partial(e))^{-1} \triangleright' \left\{ \psi_1(e)^{-1}, \hat{s}(g)^{-1} \right\}^{-1} \right) \delta \left(\phi_1(g) \triangleright \left\{ \hat{s}(g), \hat{s}(\partial(e))^{-1} \psi_1(e)^{-1} \right\} (\phi_1(g)(\partial \circ \hat{s})(g)) \triangleright \hat{t}(e) \right) \\
&= \phi_1(g) \triangleright \psi_1(e) \phi_1(g\partial(e)g^{-1}) \triangleright \hat{s}(g) \phi_1(g) \triangleright \hat{s}(\partial(e)) \phi_1(g) \triangleright \hat{s}(g)^{-1} \\
&\quad \left(\phi_1(g) \triangleright \left(\hat{s}(g) \hat{s}(\partial(e))^{-1} \triangleright' \left\{ \psi_1(e)^{-1}, \hat{s}(g)^{-1} \right\}^{-1} \right) \phi_1(g) \triangleright \left\{ \hat{s}(g), \hat{s}(\partial(e))^{-1} \psi_1(e)^{-1} \right\} (\phi_1(g)(\partial \circ \hat{s})(g)) \triangleright \hat{t}(e) \right) \\
&= \phi_2(g) \triangleright \psi_2(e),
\end{aligned}$$

where in the last equality we used equation 2 of definition 4. It is also instructive to prove directly that $\mu_2: L \rightarrow L'$ is a G -equivariant group morphism. If $k, l \in L$ we have (we use (133)):

$$\begin{aligned}
\mu_2(lk) &= \mu_1(lk) \Pi(1, 1)^{-1} \hat{t}(\delta(lk)) \\
&= \mu_1(lk) \Pi(1, 1)^{-2} (\hat{s}([1])^{-1} \triangleright' \left(\left\{ \delta(\mu_1(k))^{-1}, \hat{s}([1])^{-1} \right\}^{-1} \delta(\mu_1(k)^{-1}) \triangleright' \hat{t}(\delta(l)) \right) \hat{t}(\delta(k)) \\
&= \mu_1(lk) \Pi(1, 1)^{-1} \left\{ \delta(\mu_1(k))^{-1}, \hat{s}([1])^{-1} \right\}^{-1} \delta(\mu_1(k)^{-1}) \triangleright' \hat{t}(\delta(l)) \Pi(1, 1)^{-1} \hat{t}(\delta(k)) \\
&= \mu_1(lk) \Pi(1, 1)^{-1} [\delta(\mu_1(k))^{-1}, \delta(\Pi(1, 1))]^{-1} \mu_1(k)^{-1} \hat{t}(\delta(l)) \mu_2(k) = \mu_2(l) \mu_2(l).
\end{aligned}$$

Let us now prove that $\mu_2(g \triangleright k) = \phi_2(g) \triangleright \mu_2(k)$, if $k \in L$ and $g \in G$ (we use equation 4 of definition 4, the second Peiffer relation and equation (133)):

$$\begin{aligned}
\mu_2(g \triangleright k) &= \mu_1(g \triangleright k) \Pi(1, 1)^{-1} \hat{t}(\delta(g \triangleright k)) \\
&= \mu_1(g \triangleright k) \Pi(1, 1)^{-1} \left(\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(1, g^{-1}) \Pi(g, g^{-1}) \right)^{-1} \phi_1(g) \triangleright \left(\hat{s}(g) \hat{s}(1)^{-1} \triangleright' \left\{ \psi_1(\delta(k)^{-1}), \hat{s}(g)^{-1} \right\}^{-1} \right) \\
&\quad \phi_1(g) \triangleright \left\{ \hat{s}(g), \hat{s}(1)^{-1} \psi_1(\delta(k))^{-1} \right\} \left(\phi_1(g) (\partial \circ \hat{s})(g) \right) \triangleright \hat{t}(\delta(k)) \\
&= \mu_1(g \triangleright k) \Pi(1, 1)^{-1} \left(\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(1, g^{-1}) \Pi(g, g^{-1}) \right)^{-1} \phi_1(g) \triangleright \left(\hat{s}(g) \hat{s}(1)^{-1} \hat{s}(g)^{-1} \triangleright' \left\{ \psi_1(\delta(k)^{-1}), \hat{s}(g) \right\} \right) \\
&\quad \phi_1(g) \triangleright \left(\left(\hat{s}(g) \hat{s}(1)^{-1} \hat{s}(g)^{-1} \triangleright' \left\{ \hat{s}(g), \psi_1(\delta(k))^{-1} \right\} \right) \left\{ \hat{s}(g), \hat{s}(1)^{-1} \right\} \right) \left(\phi_1(g) (\partial \circ \hat{s})(g) \right) \triangleright \hat{t}(\delta(k)) \\
&= \mu_1(g \triangleright k) \Pi(1, 1)^{-1} \left(\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(1, g^{-1}) \Pi(g, g^{-1}) \right)^{-1} \phi_1(g) \triangleright \left(\hat{s}(g) \hat{s}(1)^{-1} \hat{s}(g)^{-1} \triangleright' \psi_1(k)^{-1} \right) \\
&\quad \phi_1(g) \triangleright \left(\left(\hat{s}(g) \hat{s}(1)^{-1} \hat{s}(g)^{-1} \triangleright' (\partial(\hat{s}(g)) \triangleright \psi_1(k)) \right) \phi_1(g) \triangleright \left\{ \hat{s}(g), \hat{s}(1)^{-1} \right\} \right) \left(\phi_1(g) (\partial \circ \hat{s})(g) \right) \triangleright \hat{t}(\delta(k)) \\
&= \mu_1(g \triangleright k) \Pi(1, 1)^{-1} \left(\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(1, g^{-1}) \Pi(g, g^{-1}) \right)^{-1} \phi_1(g) \triangleright \left\{ \hat{s}(g), \hat{s}(1)^{-1} \right\} \\
&\quad \phi_1(g) \triangleright \left(\partial(\hat{s}(g)) \triangleright \hat{s}(1) \right) \triangleright' \left(\psi_1(k)^{-1} \partial(\hat{s}(g)) \triangleright \psi_1(k) \right) \left(\phi_1(g) (\partial \circ \hat{s})(g) \right) \triangleright \hat{t}(\delta(k)) \\
&= \mu_1(g \triangleright k) \Pi(1, 1)^{-1} \left(\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(1, g^{-1}) \Pi(g, g^{-1}) \right)^{-1} \phi_1(g) \triangleright \left\{ \hat{s}(g), \hat{s}(1)^{-1} \right\} \\
&\quad \phi_1(g) \triangleright \left(\partial(\hat{s}(g)) \triangleright \Pi(1, 1)^{-1} \psi_1(k)^{-1} \right) \phi_2(g) \triangleright \psi_2(k) \\
&= \mu_1(g \triangleright k) \Pi(1, 1)^{-1} \left(\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(1, g^{-1}) \Pi(g, g^{-1}) \right)^{-1} \phi_1(g) \triangleright (\hat{s}(g) \triangleright' \Pi(1, 1)) \\
&\quad \phi_1(g) \triangleright \left(\psi_1(k)^{-1} \right) \phi_2(g) \triangleright \psi_2(k).
\end{aligned}$$

Now the last term is equal to $\phi_2(g) \triangleright \psi_2(k)$, since:

$$\Pi(1, 1)^{-1} \Pi(g, g^{-1})^{-1} \Pi(1, g^{-1})^{-1} \Pi(g, g^{-1}) \Pi(1, 1) \phi_1(g) \triangleright (\hat{s}(g) \triangleright' \Pi(1, 1)) = 1. \quad (145)$$

To see this note that, in the free group $\mathcal{F}^{\text{group}}(G)$:

$$[g] [1, 1] [g]^{-1} [1, 1] [g, g^{-1}] [1, g^{-1}]^{-1} [g, g^{-1}]^{-1} [1, 1]^{-1} = \emptyset.$$

Then, by lemma 56 and equations (139) to (143), we also know that we have a group map:

$$\mathcal{F}^{\text{group}}(G) \supset \ker(p) \rightarrow E' \ltimes_* (E' \ltimes L'),$$

where, by definition:

$$[g] [h, i] [g]^{-1} \mapsto (\phi_1(g), s(g)) \bullet (1, \delta(\Pi(h, i)^{-1}, \Pi(h, i))),$$

where \bullet is the lifted action of \mathcal{A} . This implies equation (145).

4.1.4 Composition and inverses of lax homotopies

We now freely use the notation of 3.3.1 and 3.3.4, as well as definition 55.

Theorem 58 Consider 2-crossed modules $\mathcal{A} = \left(L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \} \right)$ and $\mathcal{A}' = \left(L' \xrightarrow{\delta} E' \xrightarrow{\partial} G', \triangleright, \{, \} \right)$. Given lax homotopies of 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$, say:

$$f = (\mu, \psi, \phi) \xrightarrow{(f, \hat{s}, \hat{t}, \Pi)} f' = (\mu', \psi', \phi') \xrightarrow{(f', \hat{s}', \hat{t}', \Pi')} f'' = (\mu'', \psi'', \phi''),$$

the explicit form of their concatenation, denoted by $(f, \hat{s}, \hat{t}, \Pi) \hat{\otimes} (f', \hat{s}', \hat{t}', \Pi')$, is:

$$f \xrightarrow{(f, \hat{s} \hat{\otimes} \hat{s}', \hat{t} \hat{\otimes} \hat{t}', \Pi \hat{\otimes} \Pi')} f'',$$

where

$$(\hat{s} \hat{\otimes} \hat{s}')(g) = \hat{s}(g) \hat{s}'(g), \text{ for each } g \in G, \quad (146)$$

$$(\hat{t} \hat{\otimes} \hat{t}')(g) = \hat{s}'(\partial(e))^{-1} \triangleright' \hat{t}(e) \hat{t}'(e), \text{ for each } e \in E, \quad (147)$$

and where

$$(\Pi \hat{\otimes} \Pi')(g, h) = \Theta^{(\hat{s}, \hat{s}')}([gh], [g], [h]) \Pi'(g, h) \Pi(g, h), \quad (148)$$

where $\Theta^{(\hat{s}, \hat{s}')}$ was defined in (105).

Moreover, the inverse of $f = (\mu, \psi, \phi) \xrightarrow{(f, \hat{s}, \hat{t}, \Pi)} f' = (\mu', \psi', \phi')$ is

$$f' = (\mu', \psi', \phi') \xrightarrow{(f', \bar{s}, \bar{t}, \bar{\Pi})} f = (\mu, \psi, \phi),$$

where if $g, h \in G$ and $e \in E$:

$$\begin{aligned} \bar{s}(g) &= \hat{s}(g)^{-1}, \\ \bar{t}(e) &= \hat{s}(\partial(e)) \triangleright' \hat{t}(e)^{-1}, \\ \bar{\Pi}(g, h) &= \Theta^{(\hat{s}, \bar{s})}([gh], [g], [h])^{-1} \Pi(g, h)^{-1}. \end{aligned} \quad (149)$$

Proof. As far as the concatenation of lax homotopies is concerned, we just need to consider the corresponding chain of strict homotopies, given by the previous theorem:

$$f \circ \text{proj} \xrightarrow{(f \circ \text{proj}, s, t)} f' \circ \text{proj} \xrightarrow{(f' \circ \text{proj}_1, s', t')} f'' \circ \text{proj},$$

and look at the construction of their concatenation in 3.3.1, noting that the underlying set of G is a free (chosen) basis of $\mathcal{F}^{\text{group}}(G)$.

We know that $s([g]) = \hat{s}(g)$, $s'([g]) = \hat{s}'(g)$ and $(s \otimes s')([g]) = s([g])s'([g])$ for each $g \in G$. Thus $(\hat{s} \hat{\otimes} \hat{s}')(g) = (s \otimes s')([g]) = \hat{s}(g)\hat{s}'(g)$ for each $g \in G$.

Analogously if $e \in E$ then $\hat{t}(e) = t(e, [\partial(e)])$, $\hat{t}'(e) = t'(e, [\partial(e)])$ and $(\hat{t} \hat{\otimes} \hat{t}')(e) = (t \otimes t')(e, [\partial(e)])$, and

$$\begin{aligned} (t \otimes t')(e, [\partial(e)]) &= \omega^{(s, s')}(\partial'(e, [\partial(e)])) s'(\partial'(e, [\partial(e)]))^{-1} \triangleright' t((e, [\partial(e)])) t'((e, [\partial(e)])) \\ &= \omega^{(s, s')}([\partial(e)]) s'([\partial(e)])^{-1} \triangleright' t((e, [\partial(e)])) t'((e, [\partial(e)])) \\ &= s'([\partial(e)])^{-1} \triangleright' t((e, [\partial(e)])) t'((e, [\partial(e)])) = \hat{s}'(\partial(e))^{-1} \triangleright' \hat{t}(e) \hat{t}'(e), \end{aligned}$$

where we used remark 37. Thus (147) follows.

Recall that given $g, h \in G$ then $\Pi(g, h) = t(1, [g, h])$ and $\Pi'(g, h) = t'(1, [g, h])$. We have:

$$\begin{aligned} (\Pi \hat{\otimes} \Pi')(g, h) &= (t \otimes t')(1, [g, h]) = \omega^{(s, s')}(\partial'(1, [g, h])) s'(\partial'(1, [g, h]))^{-1} \triangleright' t(1, [g, h]) t'(1, [g, h]) \\ &= \omega^{(s, s')}([gh]^{-1}[g][h]) s'([gh]^{-1}[g][h])^{-1} \triangleright' t(1, [g, h]) t'(1, [g, h]) \\ &= \omega^{(s, s')}([gh]^{-1}[g][h]) \delta(\Pi'(g, h)) \triangleright' t(1, [g, h]) t'(1, [g, h]) \\ &= \omega^{(s, s')}([gh]^{-1}[g][h]) \Pi'(g, h) \Pi(g, h) \\ &= \Theta^{(s, s')}([gh]^{-1}, [g], [h]) \Pi'(g, h) \Pi(g, h) = \Theta^{(\hat{s}, \hat{s}')}([gh]^{-1}, [g], [h]) \Pi'(g, h) \Pi(g, h), \end{aligned}$$

where we used remark 37 again, and noted $\hat{s}(g) = s([g])$ and $\hat{s}'(g) = s'([g])$, for each $g \in G$.

Inverses are handled in the same way. For instance if $g, h \in G$ (we use (132)):

$$\begin{aligned} \bar{\Pi}(g, h) &= \bar{t}(1, [g, h]) = (\omega^{(s, \bar{s})}(\partial'(1, [g, h]))^{-1} s(\partial'(1, [g, h])) \triangleright' t(1, [g, h])^{-1} \\ &= \omega^{(s, \bar{s})}([gh]^{-1}[g][h])^{-1} s([gh]^{-1}[g][h]) \triangleright' \Pi(g, h)^{-1} \\ &= \Theta^{(s, \bar{s})}([gh], [g], [h])^{-1} \delta(\Pi(g, h))^{-1} \triangleright' \Pi(g, h)^{-1} \\ &= \Theta^{(s, \bar{s})}([gh], [g], [h])^{-1} \Pi(g, h)^{-1} = \Theta^{(\hat{s}, \bar{s})}([gh], [g], [h])^{-1} \Pi(g, h)^{-1}. \end{aligned}$$

■

4.1.5 Lax 2-fold homotopy

We now discuss lax 2-fold homotopy. Consider 2-crossed modules $\mathcal{A} = (L \xrightarrow{\delta} E \xrightarrow{\partial} G, \triangleright, \{, \})$ and $\mathcal{A}' = (L' \xrightarrow{\delta'} E' \xrightarrow{\partial'} G', \triangleright', \{, \})$. Given two lax homotopies between the 2-crossed module maps $f, f': \mathcal{A} \rightarrow \mathcal{A}'$, say:

$$\begin{array}{ccc} & (f, \hat{s}', \hat{t}', \Pi') & \\ f & \xrightarrow{\quad} & f' \\ & (f, \hat{s}, \hat{t}, \Pi) & \end{array}$$

a lax 2-fold homotopy

$$\begin{array}{ccc} & (f, \hat{s}', \hat{t}', \Pi') & \\ f & \xrightarrow{\quad} & f' \\ & \uparrow (f, \hat{s}, \hat{t}, \Pi, \hat{k}) & \\ & (f, \hat{s}, \hat{t}, \Pi) & \end{array}$$

is given by a map $\hat{k}: G \rightarrow L'$, without any restrictions, apart from that it should relate the two lax homotopies as in (150), (151) and (152), below. This is because to \hat{k} we associate the unique (strict) quadratic $(f \circ \text{proj}, s, t)$ 2-derivation $k: \mathcal{F}^{\text{group}}(G) \rightarrow L'$ such that $k([g]) = \hat{k}(g)$ for each $g \in G$. Therefore we must have:

$$\hat{s}'(g) = \hat{s}(g)\delta(\hat{k}(g)), \text{ for each } g \in G, \quad (150)$$

and for each $e \in E$:

$$\begin{aligned} \hat{t}'(e) &= t'((e, [\partial(e)])) = k(\partial'(e, [\partial(e)]))^{-1} t((e, [\partial(e)])) = k([\partial(e)])^{-1} t((e, [\partial(e)])) \\ &= \hat{k}(\partial(e))^{-1} \hat{t}(e). \end{aligned} \quad (151)$$

Also, by using equation (100) in lemma 31, for each $g, h \in G$:

$$\begin{aligned} \Pi'(g, h) &= t'((1, [g, h])) = k(\partial'((1, [g, h])))^{-1} t((1, [g, h])) = k([gh]^{-1} [g][h])^{-1} \Pi(g, h) \\ &= \left(\Xi^{(\phi, s, k)}([gh], [g], [h]) \right)^{-1} \Pi(g, h). \end{aligned} \quad (152)$$

Thus, by using equation (100):

$$\Pi'(1, 1) = \left(\Xi^{(\phi, s, k)}([1], [1], [1]) \right)^{-1} \Pi(1, 1) = k([1])^{-1} \Pi(1, 1) = \hat{k}(1)^{-1} \Pi(1, 1).$$

In particular, we can prove directly (this follows however by construction) that, if (150), (151) and (152) are satisfied, then if $f \xrightarrow{(f, \hat{s}, \hat{t}, \Pi)} f'$, we must also have that $f \xrightarrow{(f, \hat{s}', \hat{t}', \Pi')} f'$, by using (144).

If we have a chain of 2-fold homotopies $(f, \hat{s}, \hat{t}, \Pi) \xrightarrow{(f, \hat{s}, \hat{t}, \Pi, \hat{k})} (f, \hat{s}', \hat{t}', \Pi') \xrightarrow{(f, \hat{s}', \hat{t}', \Pi', \hat{k}')} (f, \hat{s}'', \hat{t}'', \Pi'')$, diagrammatically:

$$\begin{array}{ccc} & (f, \hat{s}'', \hat{t}'', \Pi'') & \\ & \uparrow (f, \hat{s}', \hat{t}', \Pi', \hat{k}') & \\ f & \xrightarrow{(f, \hat{s}', \hat{t}', \Pi')} & f' \\ & \uparrow (f, \hat{s}, \hat{t}, \Pi, \hat{k}) & \\ & (f, \hat{s}, \hat{t}, \Pi) & \end{array}$$

Then their vertical concatenation is given by the map $\hat{k} \hat{\diamond} \hat{k}': G \rightarrow L$ such that

$$(\hat{k} \hat{\diamond} \hat{k}')(g) = \hat{k}(g)\hat{k}'(g),$$

for each $g \in G$. Even though this follows by construction, let us see directly that $\hat{k}\hat{s}\hat{k}'$ does connect $(f, \hat{s}, \hat{t}, \Pi)$ and $(f, \hat{s}'', \hat{t}'', \Pi'')$. At the (the only non trivial) level of the maps $\Pi, \Pi'': G \times G \rightarrow L'$ this follows from:

$$\begin{aligned}\Pi''(g, h) &= \left(\Xi^{(\phi, s', k')}([gh], [g], [h]) \right)^{-1} \Pi'(g, h) = \left(\Xi^{(\phi, s', k')}([gh], [g], [h]) \right)^{-1} \left(\Xi^{(\phi, s, k)}([gh], [g], [h]) \right)^{-1} \Pi(g, h) \\ &\doteq (k([gh]^{-1}[g][h]) k'([gh]^{-1}[g][h]))^{-1} \Pi(g, h) = ((k \diamond k')([gh]^{-1}[g][h]))^{-1} \Pi(g, h) \\ &\doteq \left(\Xi^{(\phi, s, k \diamond k')}([gh], [g], [h]) \right)^{-1} \Pi(g, h),\end{aligned}$$

where we used (100) again, as well as lemma 45.

Suppose that we have a 2-fold lax homotopy say $(f, \hat{s}, \hat{t}, \Pi) \xrightarrow{(f, \hat{s}, \hat{t}, \Pi, \hat{k})} (f, \hat{s}', \hat{t}', \Pi')$, which we write as:

$$\begin{array}{ccc} & (f, \hat{s}', \hat{t}', \Pi) & \\ f & \begin{array}{c} \nearrow \\ \uparrow (f, \hat{s}, \hat{t}, \Pi, \hat{k}) \\ \searrow \end{array} & f' \\ & (f, \hat{s}, \hat{t}, \Pi) & \end{array}$$

and that we also have a lax homotopy:

$$f' = (\mu', \psi', \phi') \xrightarrow{(f', \hat{s}'', \hat{t}'', \Pi'')} f'' = (\mu'', \psi'', \phi''),$$

so what we have diagrammatically is:

$$\begin{array}{ccccc} & (f, \hat{s}', \hat{t}', \Pi') & & & \\ f & \begin{array}{c} \nearrow \\ \uparrow (f, \hat{s}, \hat{t}, \Pi, \hat{k}) \\ \searrow \end{array} & f' & \xrightarrow{(f', \hat{s}'', \hat{t}'', \Pi'')} & f'' \\ & (f, \hat{s}, \hat{t}, \Pi) & & & \end{array}$$

The whiskering:

$$(f, \hat{s}, \hat{t}, \Pi, \hat{k}) \hat{\otimes} (f', \hat{s}'', \hat{t}'', \Pi'') = (f, \hat{s} \hat{\otimes} \hat{s}'', \hat{t} \hat{\otimes} \hat{t}'', \Pi \hat{\otimes} \Pi'', k \hat{\otimes} s'')$$

is given by the map $\hat{k} \hat{\otimes} \hat{s}'': G \rightarrow L'$, which has the form, for each $g \in G$:

$$(\hat{k} \hat{\otimes} \hat{s}'')(g) = \hat{s}''(g)^{-1} \triangleright' \hat{k}(g).$$

By construction we have:

$$\begin{array}{ccc} & (f, \hat{s}' \hat{\otimes} \hat{s}'', \hat{t}' \hat{\otimes} \hat{t}'', \Pi' \hat{\otimes} \Pi'') & \\ f & \begin{array}{c} \nearrow \\ \uparrow (f, \hat{s} \hat{\otimes} \hat{s}'', \hat{t} \hat{\otimes} \hat{t}'', \Pi \hat{\otimes} \Pi'', \hat{k} \hat{\otimes} \hat{s}'') \\ \searrow \end{array} & f' \\ & (f, \hat{s} \hat{\otimes} \hat{s}'', \hat{t} \hat{\otimes} \hat{t}'', \Pi \hat{\otimes} \Pi'') & \end{array}$$

Similarly, suppose that we have 2-crossed module maps $f, f': \mathcal{A}' \rightarrow \mathcal{A}$, lax homotopies $(f, \hat{s}, \hat{t}, \Pi)$ and $(f, \hat{s}', \hat{t}', \Pi')$, connecting f and f' , a 2-fold homotopy $(f, \hat{s}, \hat{t}, \Pi, \hat{k})$, connecting them, and also a lax homotopy $f'' \xrightarrow{(f'', \hat{s}'', \hat{t}'', \Pi'')} f$, diagrammatically:

$$\begin{array}{ccc} & (f, \hat{s}', \hat{t}', \Pi') & \\ f'' & \xrightarrow{(f'', \hat{s}'', \hat{t}'', \Pi'')} f & \begin{array}{c} \nearrow \\ \uparrow (f, \hat{s}, \hat{t}, \Pi, \hat{k}) \\ \searrow \end{array} & f' \\ & (f, \hat{s}, \hat{t}, \Pi) & \end{array}$$

The whiskering:

$$(f'', \hat{s}'', \hat{t}'', \Pi'') \hat{\otimes} (f, \hat{s}, \hat{t}, \Pi, \hat{k}) = (f'', \hat{s}'' \hat{\otimes} \hat{s}, \hat{t}'' \hat{\otimes} \hat{t}, \Pi'' \hat{\otimes} \Pi, \hat{s}'' \hat{\otimes} \hat{k}),$$

is such that, for each $g \in G$ we have $(\hat{s}'' \hat{\otimes} \hat{k})(g) = \hat{k}(g)$. By construction:

$$\begin{array}{ccc}
 & (f'', s'' \otimes s', t'' \otimes t'') & \\
 & \curvearrowright & \\
 f'' & \uparrow (f'', s'' \hat{\otimes} \hat{s}, \hat{t}'' \hat{\otimes} \hat{t}, \Pi'' \hat{\otimes} \Pi, \hat{s}'' \hat{\otimes} \hat{k}) & f' \\
 & \curvearrowleft & \\
 & (f'', \hat{s}'' \hat{\otimes} \hat{s}, \hat{t}'' \hat{\otimes} \hat{t}, \Pi'' \hat{\otimes} \Pi) &
 \end{array}$$

By definition (since this is simply an unpacked version of definition 55), it follows:

Theorem 59 *Let \mathcal{A} and \mathcal{A}' be 2-crossed modules. There exists a 2-groupoid $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$, whose objects are the (strict) 2-crossed module maps $\mathcal{A} \rightarrow \mathcal{A}'$, the morphisms are the (pointed) lax homotopies between 2-crossed module maps, and the 2-morphisms are the (pointed) lax 2-fold homotopies between lax homotopies, whose explicit descriptions, and various concatenations, are made explicit in 4.1.3, 4.1.4 and 4.1.5.*

4.2 Composition of lax homotopies with strict 2-crossed module maps

Theorem 60 *Let \mathcal{A} , \mathcal{A}' and \mathcal{A}'' be 2-crossed modules. Let $f, f': \mathcal{A} \rightarrow \mathcal{A}'$ be 2-crossed module maps. Let also $h = (\mu, \psi, \phi): \mathcal{A}' \rightarrow \mathcal{A}''$ be another 2-crossed module map. If we have a lax 2-crossed module homotopy $(f, \hat{s}, \hat{t}, \Pi)$ connecting f and f' , then*

$$(h \circ f, \psi \circ \hat{s}, \mu \circ \hat{t}, \mu \circ \Pi) \doteq h \circ (f, \hat{s}, \hat{t}, \Pi)$$

is a lax homotopy connecting $g \circ f$ and $g \circ f'$.

Proof. Equations (139) to (144) are satisfied since h preserves all 2-crossed module operations, strictly. ■

Analogously:

Theorem 61 *Let \mathcal{A} , \mathcal{A}' and \mathcal{A}'' be 2-crossed modules. Let $f, f': \mathcal{A} \rightarrow \mathcal{A}'$ be 2-crossed module maps. Let also $h' = (\mu', \psi', \phi'): \mathcal{A}'' \rightarrow \mathcal{A}$ be a 2-crossed module morphism. If we have a lax 2-crossed module homotopy $(f, \hat{s}, \hat{t}, \Pi)$ connecting f and f' . Then*

$$(f \circ h', \hat{s} \circ \phi', \hat{t} \circ \psi', \Pi \circ (\phi' \times \phi')) \doteq (f, \hat{s}, \hat{t}, \Pi) \circ h'$$

is a lax homotopy connecting $f \circ h'$ and $f' \circ h''$.

The operators defined in theorems 60 and 61 will be called composition operators.

Theorem 62 *The composition operators preserve concatenations and inverses of lax homotopies.*

Proof. Immediate from the explicit form of the concatenations and inverses of lax homotopies, and the fact that we only compose homotopies with strict 2-crossed module morphisms. ■

We thus have a sesquicategory [44], whose objects are the 2-crossed modules, the morphisms are the 2-crossed module maps, and the 2-morphisms are the lax homotopies between them. It is important to note that this is not a 2-category, since the interchange law does not hold in general.

The composition operators are also defined, in the obvious way, and with the obvious properties, for lax 2-fold homotopies and strict 2-crossed module maps. Given that for any two 2-crossed modules \mathcal{A} and \mathcal{A}' we have a 2-groupoid $\mathcal{HOM}_{\text{LAX}}(\mathcal{A}, \mathcal{A}')$, we expect that this will give a Gray category [16, 27, 28] of 2-crossed modules, (strict) 2-crossed module maps, lax homotopies and lax 2-fold homotopies.

4.3 Lax homotopy equivalence of 2-crossed modules

In this subsection we make use of subsection 4.2. Let \mathcal{A} and \mathcal{A}' be 2-crossed modules.

Definition 63 *We say that $f: \mathcal{A} \rightarrow \mathcal{A}'$ is a lax homotopy equivalence if there exists a 2-crossed module map $g: \mathcal{A}' \rightarrow \mathcal{A}$, and lax homotopies:*

$$\text{id}_{\mathcal{A}} \xrightarrow{(\text{id}_{\mathcal{A}}, \hat{s}, \hat{t}, \Pi)} g \circ f \text{ and } \text{id}_{\mathcal{A}'} \xrightarrow{(\text{id}_{\mathcal{A}'}, \hat{u}, \hat{v}, J)} f \circ g.$$

In such a case g is said to be a homotopy inverse of f .

Lemma 64 *The composition of lax homotopy equivalences is a lax homotopy equivalence.*

Proof. This result is almost immediate from the fact that we can concatenate lax homotopies between strict 2-crossed module maps. Let us give details. Let $\mathcal{A}, \mathcal{A}'$ and \mathcal{A}'' be 2-crossed modules. Suppose that $f: \mathcal{A} \rightarrow \mathcal{A}'$ and $f': \mathcal{A}' \rightarrow \mathcal{A}''$ are lax homotopy equivalences. Let us see that $f' \circ f$ is a lax homotopy equivalence. Choose inverses up to homotopy g and g' of f and g' , respectively. We thus have lax homotopies

$$\text{id}_{\mathcal{A}} \xrightarrow{(\text{id}_{\mathcal{A}}, \hat{s}, \hat{t}, \Pi)} g \circ f, \quad \text{id}_{\mathcal{A}'} \xrightarrow{(\text{id}_{\mathcal{A}'}, \hat{u}, \hat{v}, J)} f \circ g, \quad \text{id}_{\mathcal{A}'} \xrightarrow{(\text{id}_{\mathcal{A}'}, \hat{s}', \hat{t}', \Pi')} g' \circ f', \quad \text{id}_{\mathcal{A}''} \xrightarrow{(\text{id}_{\mathcal{A}''}, \hat{u}', \hat{v}', J')} f' \circ g'.$$

We prove that $g \circ g'$ is a lax homotopy inverse of $f' \circ f$. This follows by considering the concatenations below of lax homotopies:

$$\begin{aligned} \text{id}_{\mathcal{A}} &\xrightarrow{(\text{id}_{\mathcal{A}}, \hat{s}, \hat{t}, \Pi)} g \circ f = g \circ \text{id}_{\mathcal{A}'} \circ f \xrightarrow{g \circ (\text{id}_{\mathcal{A}'}, \hat{s}', \hat{t}', \Pi') \circ f} g \circ g' \circ f' \circ f, \\ \text{id}_{\mathcal{A}''} &\xrightarrow{(\text{id}_{\mathcal{A}''}, \hat{u}', \hat{v}', J')} f' \circ g' = f' \circ \text{id}_{\mathcal{A}'} \circ g' \xrightarrow{f' \circ (\text{id}_{\mathcal{A}'}, \hat{u}, \hat{v}, J) \circ g'} f' \circ f \circ g \circ g'. \end{aligned}$$

■

Proposition 65 *The class of lax homotopy equivalences of 2-crossed modules has the two-of-three property [18].*

Proof. The complete proof is analogous to the proof of the particular case above. ■

By using the composition operators, we can easily see that a retract of a lax homotopy equivalence is again a lax homotopy equivalence.

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References

- [1] Arslan U. E.; Arvasi Z.; Onarl G.: Induced Two Crossed Modules, arXiv:1107.4291 [math.AT].
- [2] Arvasi Z., Ulualan E.: On algebraic models for homotopy 3-types. J. Homotopy Relat. Struct. 1 (2006), no. 1, 1–27.
- [3] Baues H. J.: Algebraic homotopy. Cambridge Studies in Advanced Mathematics, 15. Cambridge University Press, Cambridge, 1989.
- [4] Baues H. J.: Combinatorial homotopy and 4-dimensional complexes. With a preface by Ronald Brown. de Gruyter Expositions in Mathematics, 2. Walter de Gruyter & Co., Berlin, 1991.
- [5] Berger C.: Double loop spaces, braided monoidal categories and algebraic 3-type of space. Higher homotopy structures in topology and mathematical physics (Poughkeepsie, NY, 1996), 49–66, Contemp. Math., 227, Amer. Math. Soc., Providence, RI, 1999.
- [6] Brown R.: Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems. Galois theory, Hopf algebras, and semiabelian categories, 101–130, Fields Inst. Commun., 43, Amer. Math. Soc., Providence, RI, 2004.
- [7] Brown R., Gilbert N. D.: Algebraic models of 3-types and automorphism structures for crossed modules. Proc. London Math. Soc. (3) 59 (1989), no. 1, 51–73.
- [8] Brown R., Golasiński M.: A model structure for the homotopy theory of crossed complexes. Cahiers Topologie Géom. Différentielle Catég. 30 (1989), no. 1, 61–82.
- [9] Brown R., Higgins P. J.: Tensor Products and Homotopies for ω -Groupoids and Crossed Complexes, J. Pure Appl. Algebra 47 (1987), no. 1, 1–33.

- [10] Brown R., Higgins P. J., Sivera R.: Nonabelian algebraic topology, filtered spaces, crossed complexes, cubical homotopy groupoids. EMS Tracts in Mathematics, Vol 15. 2010.
- [11] Brown R., Sivera R.: Normalisation for the fundamental crossed complex of a simplicial set. Journal of Homotopy and Related Structures, vol. 2(2), 2007, pp.49–79.
- [12] Carrasco P. and Cegarra A. M.: Group-theoretic algebraic models for homotopy types, J. Pure Appl. Algebra 75 (1991) 195–235.
- [13] Cabello J. G., Garzón A. R.: Quillen’s theory for algebraic models of n -types. Extracta Math. 9 (1994), no. 1, 42–47.
- [14] Cabello J. G., Garzón A. R.: Closed model structures for algebraic models of n -types. J. Pure Appl. Algebra 103 (1995), no. 3, 287–302.
- [15] Conduché D.: Modules croisés généralisés de longueur 2. Proceedings of the Luminy conference on algebraic K -theory (Luminy, 1983). J. Pure Appl. Algebra 34 (1984), no. 2-3, 155–178.
- [16] Crans S. E.: A tensor product for *Gray*-categories. Theory Appl. Categ. 5 (1999), No. 2, 12–69 (electronic).
- [17] Curtis E. B.: Simplicial homotopy theory. Advances in Math. 6 1971 107–209 (1971).
- [18] Dwyer W. G., Spaliński J.: Homotopy theories and model categories. Handbook of algebraic topology, 73–126, North-Holland, Amsterdam, 1995.
- [19] Everaert T., Kieboom R. W., Van der Linden T.: Model structures for homotopy of internal categories. Theory Appl. Categ. 15 (2005/06), No. 3, 66–94.
- [20] Faria Martins J.: The fundamental crossed module of the complement of a knotted surface. Trans. Amer. Math. Soc. 361 (2009), no. 9, 4593–4630.
- [21] Faria Martins J.: The fundamental 2-crossed complex of a reduced CW-complex. Homology Homotopy Appl. 13 (2011), no. 2, 129–157.
- [22] Faria Martins J., Picken R.: The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module. Differential Geom. Appl. 29 (2011), no. 2, 179–206.
- [23] Garner R.: Homomorphisms of higher categories. Adv. Math. 224 (2010), no. 6, 2269–2311.
- [24] Garzón A. R., Miranda J. G.: Homotopy theory for truncated weak equivalences of simplicial groups. Math. Proc. Cambridge Philos. Soc. 121 (1997), no. 1, 51–74.
- [25] Gohla B.: Mapping spaces of Gray-Categories. In preparation.
- [26] Goerss P. G., Jardine J. F.: Simplicial homotopy theory. Progress in Mathematics, 174. Birkhäuser Verlag, Basel, 1999.
- [27] Gordon R., Power A.J., Street R.: Coherence for tricategories. Mem. Amer. Math Soc. 117 (1995) no 558.
- [28] Gurski N.: An algebraic theory of tricategories. PhD thesis, University of Chicago, 2006. See <http://www.math.yale.edu/~mg622/tricats.pdf>
- [29] Hardie K. A.; Kamps K. H.; Kieboom, R. W.: A Homotopy 2-Groupoid of a Hausdorff Space. Appl. Categ. Structures 8 (2000), no. 1-2, 209–234.
- [30] Hatcher A.: Algebraic Topology. Cambridge University Press, 2002.
- [31] Kamps K. H.; Porter, T.: 2-groupoid enrichments in homotopy theory and algebra. K -Theory 25 (2002), no. 4, 373–409.
- [32] Lack S.: A Quillen model structure for 2-categories. K -Theory 26 (2002), no. 2, 171–205.

- [33] Lack S.: A Quillen model structure for Gray-categories. *J. K-Theory* 8 (2011), no. 2, 183–221.
- [34] Loday J.-L.: Spaces with finitely many nontrivial homotopy groups. *J. Pure Appl. Algebra* 24 (1982), no. 2, 179–202.
- [35] May J. P.: *Simplicial objects in algebraic topology*. Reprint of the 1967 original. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.
- [36] Mac Lane S.: *Categories for the working mathematician*. Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998.
- [37] Moerdijk I., Svensson J. A.: Algebraic classification of equivariant homotopy 2-types. I. *J. Pure Appl. Algebra* 89 (1993), no. 1-2, 187–216.
- [38] Mutlu A., Porter T.: Freeness conditions for 2-crossed modules and complexes. *Theory Appl. Categ.* 4 (1998), No. 8, 174–194 (electronic)
- [39] Mutlu A., Porter T.: Freeness conditions for crossed squares and squared complexes. *J. K-Theory* 20, No.4, 345–368 (2000).
- [40] Noohi B.: Notes on 2-groupoids, 2-groups, and crossed-modules. *Homology, Homotopy and Applications*, Vol. 9 (2007), no. 1, 75–106.
- [41] Porter T.: Crossed Menagerie: an introduction to crossed gadgetry and cohomology in algebra and topology. Online notes: <http://ncatlab.org/timporter/files/menagerie9.pdf>
- [42] Porter T.: n -types of simplicial groups and crossed n -cubes. *Topology* 32 (1993), no. 1, 5–24.
- [43] Quillen D.G.: *Homotopical algebra*. Lecture Notes in Mathematics, No. 43 Springer-Verlag, Berlin-New York 1967.
- [44] Street R.: Categorical structures, *Handbook of Algebra Volume 1*, 1996, Pages 529–577
- [45] Strøm A.: The homotopy category is a homotopy category. *Arch. Math. (Basel)* 23 (1972), 435–441.
- [46] Whitehead J. H. C.: On Adding Relations to Homotopy Groups, *Ann. of Math. (2)* 42, (1941), 409–428.
- [47] Whitehead J. H. C.: Note on a Previous Paper Entitled "On Adding Relations to Homotopy Groups.", *Ann. of Math. (2)* 47, (1946). 806–810.
- [48] Whitehead J. H. C.: Combinatorial Homotopy. II. *Bull. Amer. Math. Soc.* 55, (1949). 453–496.